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KEY

TO

SHELDONS'

ELEMENTS OF ALGEBRA

AND

SHELDONS'

COMPLETE ALGEBRA.

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EMBRACES:

SHELDONS' ELEMENTARY ARITHMETIC,

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SHELDONS' ELEMENTS OF ALGEBRA,

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KEY

TO

SHELDONS'

ELEMENTS OF ALGEBRA.

Art. 40.

1. $ab = 3 \times 5 = 15.$
2. $a^2bx = 9 \times 5 \times 10 = 450.$
3. $ac^2y = 3 \times 64 \times 4 = 768.$
4. $a^3bz = 27 \times 5 \times 6 = 810.$
5. $b^2xy = 25 \times 10 \times 4 = 1000.$
6. $2ab(a+b) = 2 \times 3 \times 5 \times 8 = 240.$
7. $3x^2(x-y) = 3 \times 100 \times 6 = 1800.$
8. $a(x^2+u^2) = 3 \times 181 = 543.$
9. $xy(xy-y^2) = 10 \times 4 \times (40-16) = 960.$
10. $uz(z+u) = 9 \times 6 \times 15 = 810.$
11. $(x-y)^2 = (10-4)^2 = 36.$
12. $(b^2-x)^2 = (25-10)^2 = 225.$
13. $(u-z)^3 = (9-6)^3 = 27.$
14. $b(u^2-c^2) = 5 \times (81-64) = 85.$
15. $ab(u-c)^2 = 3 \times 5 \times (9-8)^2 = 15.$
16. $3a(x+y) = 3 \times 3 \times 14 = 126.$
17. $a^2+bc = 9+(5 \times 8) = 49.$
18. $y^3z+3zu = (64 \times 6)+(3 \times 6 \times 9) = 546.$
19. $5abc-xyz = (5 \times 3 \times 5 \times 8)-(10 \times 4 \times 6) = 360.$
20. $x^2y-xy^2 = (100 \times 4)-(10 \times 16) = 240.$
21. $ab+x-yz+u = 15+10-24+9 = 10.$
22. $b(a+b) \div (y+z) = (5 \times 8) \div 10 = 4.$
23. $(x+y)(x-y) = 14 \times 6 = 84.$
24. $(a^2-c)(u^2+x^2) = (9-8) \times (81+100) = 181.$
25. $a^2+b^2-(x^2-u^2) = 9+25-(100-81) = 15.$
26. $(z^2-y^2)-(b^2-a^2) = (36-16)-(25-9) = 4.$
27. $(x^2-\{u^2-(c^2-a^2)\} = 100-\{81-(64-9)\} = 100-26 = 74.$

28. $bc-(x+\overline{u^2-c^2}) = 40-(10+\overline{81-64}) = 40-27 = 13.$

29. $\left(\frac{ax}{b}+z\right) \div y = \left(\frac{30}{5}+6\right) \div 4 = 12 \div 4 = 3.$

30. $b+\frac{c-a}{4}+u = 5+1\frac{1}{4}+9 = 15\frac{1}{4}.$

31. $\frac{ab+c-(a+x)}{b} = \frac{15+8-13}{5} = \frac{10}{5} = 2.$

32. $\frac{2ab}{y}+\frac{2xy}{c}+\frac{b+y}{u} = \frac{30}{4}+\frac{80}{8}+\frac{9}{9} = 18\frac{1}{2}.$

33. $3x\left\{\left(\frac{u^2}{a+z}-\frac{a+b}{c}\right) \div c\right\} = 30\left\{\left(\frac{81}{9}-\frac{8}{8}\right) \div 8\right\}$
$= 30 \times (8 \div 8) = 30.$

34. $\frac{c^2+x-b}{a}+\frac{(z^2+u)x}{b} = \frac{64+10-5}{3}+\frac{(36+9)\times 10}{5} = 23+90 = 113.$

35. $\frac{a+b^2+c+(x^2-c^2)}{u^2-(c^2+u)} = \frac{3+25+8+(100-64)}{81-(64+9)} = \frac{72}{8} = 9.$

36. $\frac{(a+b)(b+c)(x-c)}{a^2+y} = \frac{8\times 13\times 2}{9+4} = \frac{208}{13} = 16.$

37. $\frac{5(xyz-abc)}{bc-ax}+\frac{a^2+b^2+c^2}{3ab+y} = \frac{5\times(240-120)}{40-30}+\frac{9+25+64}{45+4} = \frac{600}{10}+\frac{98}{49}$
$= 62.$

38. $3a^2x^2y-\{2a^2xz-[(a+b)(y^2-u)]\} = 10800-\{3240-[8\times 55]\}$
$= 10800-(3240-440) = 8000.$

Art. 47.

11. Let x = price per pound.
Then $5x+35 = 95$, or $5x = 95-35$ or 60.
Whence, $x = 12$, or 12 cts.

12. Let x = price allowed per bushel.
Then $6x = \$4+20$¢, or $6x = \$4.20$. Whence, $x = 70$ cts.

13. Let x = price per ton.
Then $6x+\$5 = \$23$, or $6x = \$23-\$5$, or $\$18$. Whence, $x = \$3.$

14. Let x = cost of bat.

Then will $20+2x$ = cost of ball,

and $x+20+2x = 65$;

or $3x = 65-20$ or 45.

Whence, $x = 15$, or 15 cts., bat,

and $20+2x = 50$, or 50 cts., ball.

15. Let x = No. qts. Mary picked.

Then will $2x-7$ = No. qts. Jane picked,

and $x+2x-7 = 29$,

or $3x = 29+7$, or 36.

Whence, $x = 12$, No. qts. Mary picked,

and $2x-7 = 17$, " Jane "

16. Let x = age of son.

Then $5x-12$ = age of father,

and $x+5x-12 = 60$,

or $6x = 60+12$ or 72.

Whence, $x = 12$, age of son,

and $5x-12 = 48$, age of father.

17. Let x = cost of slate.

Then will $3x+15¢$ = cost of book,

and $x+3x+15¢ = \$1.35$,

or $4x = \$1.35-\$.15$, or $\$1.20$.

Whence, $x = 30¢$, cost of slate,

and $3x+15¢ = \$1.05$, cost of book.

18. Let x = price per yard.

Then $12x-7x = 70¢$, or $5x = 70¢$. Whence, $x = 14¢$.

19. Let x = cost of each horse.

Then $2x+\$20+\$35 = \$415$,

or $2x = \$415-\$55$, or $\$360$.

Whence, $x = \$180$.

20. Let x = cost of second.

Then $2x+\$500$ = cost of first,

and $x-\$900+2x+\$1300 = \$7900$,

or $3x = \$7900-\$400$, or $\$7500$.

Whenc $x = \$2500$, cost of second,

and $2x + \$500 = \$5500$, " first.

$\$2500 - \$900 = \$1600$, rec'd for second,

$\$5500 + \$800 = \$6300$, " first.

Art. 59.

13.
$$\begin{array}{r} ax - 4ab + bd \\ -2ax + ab + 3bd \\ -2ax + 7ab - bd \\ -3ax + 5ab + 12bd \\ \hline -6ax + 9ab + 15bd \end{array}$$

14.
$$\begin{array}{r} 3abd + 4abx - 5cx \\ 12abd - 11abx + 8cx \\ 3abd + 9abx - 12cx \\ 3abd - 15abx + 7cx \\ \hline 21abd - 13abx - 2cx \end{array}$$

15.
$$\begin{array}{r} 7b - 2a^2 - xy \\ -6b + 5a^2 + 3xy \\ b - 3a^2 \quad + 4 \\ -b + a^2 - 3xy \\ \hline b + a^2 - xy + 4 \end{array}$$

16.
$$\begin{array}{r} 3b^2 - 2a^3 + 13 \\ -2b^2 + 3a^3 - 5 + 4ab \\ 7a^3 - 3 \\ 2b^2 - a^3 \quad + ab \\ \hline 3b^2 + 7a^3 + 5 + 5ab \end{array}$$

17.
$$\begin{array}{r} ax^2 - 2y + b \\ 2ax^2 + 2y - 3b \\ 4ax^2 - y + b \\ -3ax^2 \quad + 2b \\ b \\ \hline 4ax^2 - y + 2b \end{array}$$

18.
$$\begin{array}{r} 2c^2 + a^2 + 3bc \\ 5c^2 - 3a^2 - 2bc \\ c^2 + 2a^2 - bc \\ + bc + b - 3 \\ \hline 8c^2 \quad + bc + b - 3 \end{array}$$

19.
$$\begin{array}{r} ab + a^2c - 5 \\ 3ab - 3a^2c + 7 \\ -2ab + 2a^2c - 3 \\ ab + a^2c + 5 \\ \hline 3ab + a^2c + 4 \end{array}$$

20.
$$\begin{array}{r} 3b^2 - 2a^2x + b \\ -b^2 + 3a^2x - 3b \\ b^2 - a^2x \quad + c \\ 3b^2 \quad + b - 3c \\ \hline 6b^2 \quad - b - 2c \end{array}$$

21.
$$\begin{array}{r} 2a^2 + 3 - ac \\ 3a^2 - 7 + ac \\ -5a^2 + 9 + 3ac \\ -a^2 + 4 + 3ac \\ \hline -a^2 + 9 + 6ac \end{array}$$

22.
$$\begin{array}{r} b^2c + 2 - y^2 \\ -3b^2c - 10 \quad + y^3 \\ 2b^2c - 3 + 2y^2 \\ 5 - y^2 \quad + b^2 \\ \hline -6 \quad + y^3 + b^2 \end{array}$$

23.
$$\begin{array}{r} 5a+3b-\frac{1}{2}c+d \\ 2b+c-3d \\ 7a-5b+\frac{1}{4}c+2d \\ \hline 12a \qquad +\frac{3}{4}c \end{array}$$

$$\begin{array}{r} x^2+\frac{1}{4}x-9 \qquad \\ \frac{3}{4}x+17+y \\ 3x^2-x \qquad -y \\ -8 \qquad +y^2 \\ \hline 4x^2 \qquad\qquad +y^2 \end{array}$$

25.
$$\begin{array}{r} 8m+9n+x-y \\ \frac{1}{3}m-2n+3x+4y \\ -7m \qquad -9x+3y \\ 5x-4y \\ \hline 1\frac{1}{3}m+7n \qquad +2y \end{array}$$

26.
$$\begin{array}{r} a+b+c \\ a+b-c \\ a-b+c \\ a-b-c \\ \hline 4a \qquad \end{array}$$

27.
$$\begin{array}{r} a^2+b^2+c+d \qquad \\ -6a^2+3b^2 \qquad -d+e \\ 5a^2-4b^2-3c \qquad\quad \\ \hline -2c \qquad +e \end{array}$$

28.
$$\begin{array}{r} \frac{1}{2}a+\frac{1}{6}ab+9b^2 \\ \frac{7}{8}a-\frac{2}{11}ab-15b^2 \\ \frac{5}{6}a+\frac{2}{3}ab+6b^2 \\ -a+ab \qquad \\ \hline \frac{29}{24}a+\frac{78}{66}ab \qquad \end{array}$$

29.
$$\begin{array}{r} .25ab+c+.99b \\ 3ab+2c+.01b \\ .75ab \qquad +b \\ \hline 4ab+3c+2b \end{array}$$

30.
$$\begin{array}{r} 7ax+2bx+6cx-3ay+by+4cy \\ 2ax+3bx-cx-4ay-2by \qquad \\ -5bx-5cx+5ay+by-4cy \\ \hline 9ax \qquad\qquad -2ay \qquad\qquad \end{array}$$

31.
$$\begin{array}{r} 8x^3-2x^2+7x-1 \\ -9x^3 \qquad +11x-2 \\ 19x^3+7x^2 \qquad -10 \\ 15x^3 \qquad\qquad +100 \\ \hline 28x^3+5x^2+18x+87 \end{array}$$

32.
$$\begin{array}{r} 5a+\frac{1}{3}b-\frac{1}{2}c^2 \\ \frac{1}{4}a-2b+c^2 \\ -2a+b-\frac{1}{2}c^2 \\ \hline \frac{13}{4}a-\frac{2}{3}b \qquad \end{array}$$

33.
$$\begin{array}{r} 3x+y-7z \\ -2x-6y-3z \\ 15x-3y+8z \\ -9x+4y+z \\ -2x-y-z \\ \hline 5x-5y-2z \end{array}$$

34.
$$\begin{array}{r} 3a^2bx-5ab^2x+4abx^2-6a^2b^2x \\ -2a^2bx-6ab^2x-5abx^2+3a^2b^2x \\ -a^2bx+12ab^2x \qquad\qquad\qquad \\ 8a^2bx \qquad\qquad +3abx^2 \qquad\quad \\ \hline 8a^2bx+ab^2x+2abx^2-3a^2b^2x \end{array}$$

35.
$$\begin{array}{r} 4\frac{1}{2}x^2y-3\frac{2}{3}xy^2+17x^2y^2 \\ -8\frac{3}{4}x^2y+18\frac{5}{6}xy^2-21\frac{1}{2}x^2y^2 \\ 8\frac{1}{4}x^2y-8\frac{1}{2}xy^2+4\frac{3}{4}x^2y^2 \\ -4x^2y-2\frac{2}{3}xy^2+2\frac{3}{4}x^2y^2 \\ \hline -\frac{5}{6}xy^2+3x^2y^2 \end{array}$$

36.

$$\begin{array}{r}
2a^3b^2c^3-\ 3a^2b^3c^2+12a^2b^3c^3\\
-19a^3b^2c^3+15a^2b^3c^2-14a^2b^3c^3+18a^3b^3c^3+5a^3b^3c^2\\
17a^3b^2c^3-12a^2b^3c^2+13a^2b^3c^3\\
-11a^2b^3c^3\\
\hline
18a^3b^3c^3+5a^3b^3c^2
\end{array}$$

37.

$$\begin{array}{r}
7x^2-3xy-\ x\\
3x^2\quad +\ 3x-\ y^2\\
-2x^2+4xy\quad +5y^2-y\\
-7xy+\ 9x-\ y^2\\
6xy-11x\quad +y\\
\hline
8x^2\quad\quad +3y^2
\end{array}$$

38.

$$\begin{array}{r}
\tfrac{3}{4}x^2-2x+\ \tfrac{1}{3}\\
-x^2+\ x-\ 2\\
2x^2\quad -\ \tfrac{1}{2}-5\tfrac{1}{2}xy\\
-2\tfrac{3}{4}x^2+2x+1\tfrac{1}{6}+8\tfrac{1}{4}xy\\
\hline
-x^2+\ x-\ 1+2\tfrac{3}{4}xy
\end{array}$$

Art. 67.

31.

$$\begin{array}{rl}
\tfrac{1}{2}a+b-\tfrac{3}{4}c & \\
\tfrac{1}{3}a\quad +\tfrac{1}{2}c & \\
-\ a-b+\ c & \\
\hline
-\tfrac{1}{6}a\quad +\tfrac{3}{4}c & \text{Minuend.}\\
-\tfrac{5}{6}a\quad +\tfrac{1}{4}c & \text{Subtrahend.}\\
\hline
\tfrac{2}{3}a\quad +\tfrac{1}{2}c & \text{Difference.}
\end{array}$$

32.

$$\begin{array}{rl}
.05x-\ .1y+\ z & \\
.02x+.01y-1.2z & \\
-\ .1x+.09y-\ .8z & \\
\hline
-.03x\quad\quad -z & \text{Minuend.}\\
-.03x\quad\quad +z & \text{Subtrahend.}\\
\hline
-2z & \text{Difference.}
\end{array}$$

33.

$$\begin{array}{rl}
6ax-8by+3c & \\
4ax+5by-6c & \\
-7ax+9by+2c & \\
\hline
3ax+6by-\ c & \text{Minuend.}\\
\hline
5ax+3by-6c & \\
-9ax-6by+7c & \\
\hline
-4ax-3by+\ c & \text{Subtrahend.}\\
\hline
7ax+9by-2c & \text{Difference.}
\end{array}$$

34.

$$\begin{array}{rl}
5c-3b-2a & \\
-2c-5b+4a & \\
-6c+4b-5a & \\
\hline
-3c-4b-3a & \text{Minuend.}\\
\hline
-3c-2b+3a & \\
+2c+3b-9a & \\
\hline
-\ c+\ b-6a & \text{Subtrahend.}\\
\hline
-2c-5b+3a & \text{Difference.}
\end{array}$$

Art. 70.

1. $14+4-(8-2)=18-6=12.$

2. $10-(6+3-2)=10-7=3.$

3. $13+8-(5+12)=21-17=4.$

4. $15-(12-7+9)=15-14=1.$

5. $25-\{18-(13-4)\}=$
$25-(18-9)=25-9=16.$

6. $100-[50-\{30-(20-10)\}] = 100-[50-(30-10)] = 100-(50-20)$
$= 100-30 = 70.$

7. $3a-b-(2a-b) = 3a-b-2a+b = a.$

8. $32a+3b-(5a+17b) = 32a+3b-5a-17b = 27a-14b.$

9. $a-b+c-(a-b-c) = a-b+c-a+b+c = 2c.$

10. $3a+2b-3c-(2a-b-c) = 3a+2b-3c-2a+b+c = a+3b-2c.$

11. $13a-(5c+3x-7a-5x+3a) = 13a-5c-3x+7a+5x-3a$
$= 17a-5c+2x.$

12. $x+(y-z)-(z-2y+2x)+(3x-y) = x+y-z-z+2y-2x+3x-y$
$= 2x+2y-2z.$

13. $-8a+5b-3c-(7a-3b-2c) = -8a+5b-3c-7a+3b+2c$
$= -15a+8b-c.$

14. $3a-5c+3d-(7a-5d+8c-2e) = 3a-5c+3d-7a+5d-8c+2e$
$= -4a-13c+8d+2e.$

15. $37a-5f-(3a-2b-5c)-(-6a-4b+3h)$
$= 37a-5f-3a+2b+5c+6a+4b-3h = 40a-5f+6b+5c-3h.$

16. $a-\{b-c-(d-e)\} = a-(b-c-d+e) = a-b+c+d-e.$

17. $2a-(2b-d)-\{a-b-(2c-2d)\} = 2a-2b+d-(a-b-2c+2d)$
$= 2a-2b+d-a+b+2c-2d = a-b+2c-d.$

18. $\{a-(x-y)\}-(2a+x+y) = a-x+y-2a-x-y = -a-2x.$

19. $3a-\{b+(2a-b)-(a-b)\} = 3a-(b+2a-b-a+b)$
$= 3a-b-2a+b+a-b = 2a-b.$

20. $x-[2y+\{3z-3x-(x+y)\}]+2x-(y-3z)$
$= x-2y-\{3z-3x-x-y\}+2x-y-3z$
$= x-2y-3z+3x+x+y+2x-y-3z = 7x-2y-6z.$

21. $ab-\{(ab+ac-a)-(2a-ac)-(2ac-2a)\}$
$= ab-\{ab+ac-a-2a+ac-2ac+2a\}$
$= ab-ab-ac+a+2a-ac+2ac-2a = a.$

22. $2ab-3ac-\{(ab-4ac)+(4a+2b)+(ac+ab-4a-2b)\}$
$= 2ab-3ac-ab+4ac-4a-2b-ac-ab+4a+2b = 0.$

23. $ax+b^2-(d-c)-\{2ax-d+b^2-(y^2+ax)\}$

$= ax+b^2-d+c-(2ax-d+b^2-y^2-ax)$

$= ax+b^2-d+c-2ax+d-b^2+y^2+ax = c+y^2.$

24. $\{(a-2b+ab)-(a-b+c)\}-\{a-(a-b+ab)\}$

$= \{a-2b+ab-a+b-c\}-\{a-a+b-ab\}$

$= a-2b+ab-a+b-c-a+a-b+ab = -2b+2ab-c.$

Art. 83.

1.
$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\quad\;\, +ab+b^2} \\ a^2+2ab+b^2 \end{array}$$

2.
$$\begin{array}{l} 3x+2y \\ \underline{2x+3y} \\ 6x^2+4xy \\ \underline{\qquad +9xy+6y^2} \\ 6x^2+13xy+6y^2 \end{array}$$

3.
$$\begin{array}{l} 3ab+4b^2 \\ \underline{2ab-3b^2} \\ 6a^2b^2+8ab^3 \\ \underline{\qquad\quad -9ab^3-12b^4} \\ 6a^2b^2-ab^3-12b^4 \end{array}$$

4.
$$\begin{array}{l} a+b-2c \\ \underline{2a-b} \\ 2a^2+2ab-4ac \\ \underline{\qquad -ab\qquad -b^2+2bc} \\ 2a^2+ab-4ac-b^2+2bc \end{array}$$

5.
$$\begin{array}{l} x^2+y^2+z \\ \underline{x^2-y^2} \\ x^4+x^2y^2+x^2z \\ \underline{\quad\; -x^2y^2\qquad -y^4-y^2z} \\ x^4\qquad +x^2z-y^4-y^2z \end{array}$$

6.
$$\begin{array}{l} a^2-ab+b^2 \\ \underline{a^2+ab-b^2} \\ a^4-a^3b+a^2b^2 \\ \quad\;\; +a^3b-a^2b^2+ab^3 \\ \underline{\qquad\qquad -a^2b^2+ab^3-b^4} \\ a^4\qquad\quad -a^2b^2+2ab^3-b^4 \end{array}$$

7.
$$\begin{array}{l} a^2-ab+2b^2 \\ \underline{a^2+ab+2b^2} \\ a^4-a^3b+2a^2b^2 \\ \quad\;\; +a^3b-a^2b^2+2ab^3 \\ \underline{\qquad\qquad +2a^2b^2-2ab^3+4b^4} \\ a^4\qquad\quad +3a^2b^2\qquad\quad +4b^4 \end{array}$$

8.
$$\begin{array}{l} a^3-4a^2+11a-24 \\ \underline{a^2+4a+5} \\ a^5-4a^4+11a^3-24a^2 \\ \quad\;\; +4a^4-16a^3+44a^2-96a \\ \underline{\qquad\qquad +5a^3-20a^2+55a-120} \\ a^5\qquad\qquad\qquad\qquad\quad -41a-120 \end{array}$$

9.
$$\begin{array}{l} 2x^3+4x^2+8x+16 \\ \underline{3x-6} \\ 6x^4+12x^3+24x^2+48x \\ \underline{\qquad -12x^3-24x^2-48x-96} \\ 6x^4\qquad\qquad\qquad\qquad\quad -96 \end{array}$$

10.

$$\begin{array}{l}
27x^3+9x^2y+3xy^2+y^3 \\
3x-y \\
\hline
81x^4+27x^3y+9x^2y^2+3xy^3 \\
\qquad -27x^3y-9x^2y^2-3xy^3-y^4 \\
\hline
81x^4 \qquad\qquad\qquad\qquad -y^4
\end{array}$$

11.

$$\begin{array}{l}
a^4-2a^3b+4a^2b^2-8ab^3+16b^4 \\
a+2b \\
\hline
a^5-2a^4b+4a^3b^2-8a^2b^3+16ab^4 \\
\quad +2a^4b-4a^3b^2+8a^2b^3-16ab^4+32b^5 \\
\hline
a^5 \qquad\qquad\qquad\qquad\qquad\qquad +32b^5
\end{array}$$

12.

$$\begin{array}{l}
x^4+3x^3+5x^2+7x+9 \\
x^2-2x+1 \\
\hline
x^6+3x^5+5x^4+7x^3+9x^2 \\
\quad -2x^5-6x^4-10x^3-14x^2-18x \\
\qquad\qquad +x^4+3x^3+5x^2+7x+9 \\
\hline
x^6+x^5 \qquad\qquad\qquad\qquad -11x+9
\end{array}$$

13.

$$\begin{array}{l}
a^5+3a^4+5a^3+7a^2+9a+1 \\
a^4-3a^3+4a^2-4a+4 \\
\hline
a^9+3a^8+5a^7+7a^6+9a^5+a^4 \\
\quad -3a^8-9a^7-15a^6-21a^5-27a^4-3a^3 \\
\qquad +4a^7+12a^6+20a^5+28a^4+36a^3+4a^2 \\
\qquad\qquad -4a^6-12a^5-20a^4-28a^3-36a^2-4a \\
\qquad\qquad\qquad +4a^5+12a^4+20a^3+28a^2+36a+4 \\
\hline
a^9 \qquad\qquad\qquad\qquad\qquad -6a^4+25a^3-4a^2+32a+4
\end{array}$$

14.

$$\begin{array}{l}
m^4+2m^3+3m^2+4m+5 \\
m^2-2m+1 \\
\hline
m^6+2m^5+3m^4+4m^3+5m^2 \\
\quad -2m^5-4m^4-6m^3-8m^2-10m \\
\qquad\qquad +m^4+2m^3+3m^2+4m+5 \\
\hline
m^6 \qquad\qquad\qquad\qquad\qquad -6m+5
\end{array}$$

15.

$$\begin{array}{l}
x^4+2x^3+3x^2+2x+1\\
x^2-2x+1\\
\hline
x^6+2x^5+3x^4+2x^3+x^2\\
\quad -2x^5-4x^4-6x^3-4x^2-2x\\
\qquad\qquad +x^4+2x^3+3x^2+2x+1\\
\hline
x^6 \qquad\qquad -2x^3 \qquad\qquad +1
\end{array}$$

16.

$$\begin{array}{l}
2a^4-3a^3b-5a^2b^2\\
a^3-2a^2b+3ab^2\\
\hline
2a^7-3a^6b-5a^5b^2\\
\quad -4a^6b+6a^5b^2+10a^4b^3\\
\qquad\qquad +6a^5b^2-9a^4b^3-15a^3b^4\\
\hline
2a^7-7a^6b+7a^5b^2+a^4b^3-15a^3b^4
\end{array}$$

17.

$$\begin{array}{l}
a^5-a^4b+ab^4-b^5\\
a+b\\
\hline
a^6-a^5b \qquad +a^2b^4-ab^5\\
\quad +a^5b-a^4b^2 \qquad +ab^5-b^6\\
\hline
a^6 \qquad -a^4b^2+a^2b^4 \qquad -b^6
\end{array}$$

18.

$$\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc\\
a+b+c\\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc\\
\quad -ab^2 \qquad +a^2b \qquad -abc+b^3+bc^2-b^2c\\
\qquad -ac^2 \qquad +a^2c-abc \qquad -bc^2+b^2c+c^3\\
\hline
a^3 \qquad\qquad\qquad -3abc+b^3 \qquad\qquad +c^3
\end{array}$$

19.

$$\begin{array}{l}
a^3+3a^2b+3ab^2+b^3\\
a^3-3a^2b+3ab^2-b^3\\
\hline
a^6+3a^5b+3a^4b^2+a^3b^3\\
\quad -3a^5b-9a^4b^2-9a^3b^3-3a^2b^4\\
\qquad\quad +3a^4b^2+9a^3b^3+9a^2b^4+3ab^5\\
\qquad\qquad\qquad -a^3b^3-3a^2b^4-3ab^5-b^6\\
\hline
a^6 \qquad -3a^4b^2 \qquad +3a^2b^4 \qquad -b^6
\end{array}$$

20.

$$\begin{array}{l}
27a^2-13ab+5b^2 \\
7a^2+b^2 \\ \hline
189a^4-91a^3b+35a^2b^2 \\
\qquad\qquad +27a^2b^2-13ab^3+5b^4 \\ \hline
189a^4-91a^3b+62a^2b^2-13ab^3+5b^4
\end{array}$$

21.

$$\begin{array}{l}
x^4+2x^3y+4x^2y^2+8xy^3+16y^4 \\
x-2y \\ \hline
x^5+2x^4y+4x^3y^2+8x^2y^3+16xy^4 \\
\quad -2x^4y-4x^3y^2-8x^2y^3-16xy^4-32y^5 \\ \hline
x^5 \qquad\qquad\qquad\qquad\qquad\qquad -32y^5
\end{array}$$

22.

$$\begin{array}{l}
x^5+5x^4+15x^3+15x^2+5x+1 \\
x^4-5x^3+10x^2-5x+1 \\ \hline
x^9+5x^8+15x^7+15x^6+5x^5+x^4 \\
\quad -5x^8-25x^7-75x^6-75x^5-25x^4-5x^3 \\
\qquad +10x^7+50x^6+150x^5+150x^4+50x^3+10x^2 \\
\qquad\qquad -5x^6-25x^5-75x^4-75x^3-25x^2-5x \\
\qquad\qquad\qquad +x^5+5x^4+15x^3+15x^2+5x+1 \\ \hline
x^9 \qquad\qquad -15x^6+56x^5+56x^4-15x^3 \qquad\qquad +1
\end{array}$$

23.

$$\begin{array}{l}
3x^3-15x^2y+20xy^2-12y^3 \\
x^2+5xy+3y^2 \\ \hline
3x^5-15x^4y+20x^3y^2-12x^2y^3 \\
\quad +15x^4y-75x^3y^2+100x^2y^3-60xy^4 \\
\qquad\qquad +9x^3y^2-45x^2y^3+60xy^4-36y^5 \\ \hline
3x^5 \qquad\qquad -46x^3y^2+43x^2y^3 \qquad\qquad -36y^5
\end{array}$$

24.

$$\begin{array}{l}
m^5+3m^4+4m^3+4m^2+12m+21 \\
m^4-3m^3+5m^2-7m+1 \\ \hline
m^9+3m^8+4m^7+4m^6+12m^5+21m^4 \\
\quad -3m^8-9m^7-12m^6-12m^5-36m^4-63m^3 \\
\qquad +5m^7+15m^6+20m^5+20m^4+60m^3+105m^2 \\
\qquad\qquad -7m^6-21m^5-28m^4-28m^3-84m^2-147m \\
\qquad\qquad\qquad +m^5+3m^4+4m^3+4m^2+12m+21 \\ \hline
m^9 \qquad\qquad\qquad\qquad -20m^4-27m^3+25m^2-135m+21
\end{array}$$

25.

$$\begin{array}{l}
42a^4+105a^3b+23a^2b^2+5ab^3+b^4\\
2a^2-5ab+b^2\\
\hline
84a^6+210a^5b+\ 46a^4b^2+\ 10a^3b^3+\ 2a^2b^4\\
\quad -210a^5b-525a^4b^2-115a^3b^3-25a^2b^4-5ab^5\\
\qquad\quad +\ 42a^4b^2+105a^3b^3+23a^2b^4+5ab^5+b^6\\
\hline
84a^6 \qquad -437a^4b^2 \qquad\qquad +b^6
\end{array}$$

26.

$$\begin{array}{l}
81x^4+27x^3y+9x^2y^2+3xy^3+y^4\\
3x-y\\
\hline
243x^5+81x^4y+27x^3y^2+9x^2y^3+3xy^4\\
\quad -81x^4y-27x^3y^2-9x^2y^3-3xy^4-y^5\\
\hline
243x^5 \qquad\qquad -y^5
\end{array}$$

27.

$$\begin{array}{l}
x^5+5x^4+15x^3+30x^2+24x+21\\
x^4-5x^3+10x^2-5x+1\\
\hline
x^9+5x^8+15x^7+30x^6+\ 24x^5+\ 21x^4\\
\quad -5x^8-25x^7-75x^6-150x^5-120x^4-105x^3\\
\qquad +10x^7+50x^6+150x^5+300x^4+240x^3+210x^2\\
\qquad\quad -\ 5x^6-\ 25x^5-\ 75x^4-150x^3-120x^2-105x\\
\qquad\qquad +\ x^5+\ 5x^4+\ 15x^3+\ 30x^2+\ 24x+21\\
\hline
x^9 \qquad\qquad +131x^4 \qquad +120x^2-\ 81x+21
\end{array}$$

28.

$$\begin{array}{l}
x+\ 4\\
x+10\\
\hline
x^2+\ 4x\\
\quad 10x+40\\
\hline
x^2+14x+40\\
x-7\\
\hline
x^3+14x^2+40x\\
\quad -\ 7x^2-98x-280\\
\hline
x^3+\ 7x^2-58x-280\\
x-9\\
\hline
x^4+\ 7x^3-\ 58x^2-280x\\
\quad -\ 9x^3-\ 63x^2+522x+2520\\
\hline
x^4-\ 2x^3-121x^2+242x+2520\\
x+\ 2\\
\hline
x^5-\ 2x^4-121x^3+242x^2+2520x\\
\quad 2x^4-\ 4x^3-242x^2+\ 484x+5040\\
\hline
x^5 \qquad -125x^3 \qquad +3004x+5040
\end{array}$$

29.

$$\begin{array}{l}
x+8 \\
x-8 \\ \hline
x^2+8x \\
\quad -8x-64 \\ \hline
x^2 \qquad -64 \\
x+5 \\ \hline
x^3+5x^2-64x-320 \\
x-5 \\ \hline
x^4+5x^3-64x^2-320x \\
\quad -5x^3-25x^2+320x+1600 \\ \hline
x^4 \qquad -89x^2 \qquad\qquad +1600 \\
x+3 \\ \hline
x^5+3x^4-89x^3-267x^2+1600x+4800 \\
x-3 \\ \hline
x^6+3x^5-89x^4-267x^3+1600x^2+4800x \\
\quad -3x^5-\ 9x^4+267x^3+\ 801x^2-4800x-14400 \\ \hline
x^6 \qquad -98x^4 \qquad\qquad +2401x^2 \qquad\qquad -14400
\end{array}$$

30.

$$\begin{array}{l}
a+b \\
a-b \\ \hline
a^2+ab \\
\quad -ab-b^2 \\ \hline
a^2 \qquad -b^2 \\
a^2+ab+b^2 \\ \hline
a^4+a^3b-a^2b^2-ab^3-b^4 \\
\qquad\qquad a^2b^2 \\ \hline
a^4+a^3b \qquad -ab^3-b^4 \\
a^2-ab+b^2 \\ \hline
a^6+a^5b-a^4b^2-a^3b^3-a^2b^4+ab^5 \\
\quad -a^5b+a^4b^2+a^3b^3+a^2b^4-ab^5-b^6 \\ \hline
a^6 \qquad\qquad\qquad\qquad\qquad\qquad -b^6
\end{array}$$

31.
$$\begin{array}{l}
x-4 \\
\underline{x+4} \\
x^2-4x \\
\quad\underline{+4x-16} \\
x^2 \qquad -16 \\
\underline{x+3} \\
x^3+3x^2-16x-48 \\
\underline{x-3} \\
x^4+3x^3-16x^2-48x \\
\quad\underline{-3x^3-\ 9x^2+48x+144} \\
x^4 \qquad -25x^2 \qquad +144
\end{array}$$

32.
$$\begin{array}{l}
a^2+ab+b^2 \\
\underline{a^3-a^2b+b^3} \\
a^5+a^4b+a^3b^2 \\
\quad -a^4b-a^3b^2-a^2b^3 \\
\qquad\qquad \underline{a^2b^3+ab^4+b^5} \\
a^5 \qquad\qquad\qquad +ab^4+b^5 \\
\underline{a-b} \\
a^6-a^5b+a^2b^4+ab^5 \\
\qquad\qquad \underline{-ab^5-b^6} \\
a^6-a^5b+a^2b^4 \qquad -b^6
\end{array}$$

33.
$$\begin{array}{l}
1+x \\
\underline{1+x^4} \\
1+x+x^4+x^5 \\
\underline{1-x+x^2-x^3} \\
1+x+x^4+x^5-x^2-x^6+x^3+x^7 \\
\quad\underline{-x-x^4-x^5+x^2+x^6-x^3-x^7-x^8} \\
1 \qquad\qquad\qquad\qquad\qquad\qquad -x^8
\end{array}$$

Art. 109.

1.
$$\begin{array}{l|l}
x^2+2xy+y^2 & x+y \\
\underline{x^2+\ xy} & x+y \\
\qquad xy+y^2 & \\
\qquad \underline{xy+y^2} &
\end{array}$$

2.
$$\begin{array}{l|l}
x^2-2xy+y^2 & x-y \\
\underline{x^2-\ xy} & x-y \\
\quad -\ xy+y^2 & \\
\quad \underline{-\ xy+y^2} &
\end{array}$$

3.
$$\begin{array}{l|l}
x^3-3x^2y+3xy^2-y^3 & x-y \\
\underline{x^3-\ x^2y} & x^2-2xy+y^2 \\
\quad -2x^2y+3xy^2 & \\
\quad \underline{-2x^2y+2xy^2} & \\
\qquad\qquad xy^2-y^3 & \\
\qquad\qquad \underline{xy^2-y^3} &
\end{array}$$

4.

$$\begin{array}{l|l}
x^3+x^2+4x-20 & x-2 \\ \cline{2-2}
\underline{x^3-2x^2} & x^2+3x+10 \\
\quad 3x^2+\ \ 4x & \\
\quad \underline{3x^2-\ \ 6x} & \\
\qquad\quad 10x-20 & \\
\qquad\quad \underline{10x-20} &
\end{array}$$

5.

$$\begin{array}{l|l}
x^4+2x^3-3x^2-4x+4 & x-1 \\ \cline{2-2}
\underline{x^4-\ \ x^3} & x^3+3x^2-4 \\
\quad 3x^3-3x^2 & \\
\quad \underline{3x^3-3x^2} & \\
\qquad\qquad -4x+4 & \\
\qquad\qquad \underline{-4x+4} &
\end{array}$$

6.

$$\begin{array}{l|l}
2x^3-x^2+3x-9 & 2x-3 \\ \cline{2-2}
\underline{2x^3-3x^2} & x^2+x+3 \\
\quad 2x^2+3x & \\
\quad \underline{2x^2-3x} & \\
\qquad\quad 6x-9 & \\
\qquad\quad \underline{6x-9} &
\end{array}$$

7.

$$\begin{array}{l|l}
x^4-9x^2-6xy-y^2 & x^2+3x+y \\ \cline{2-2}
\underline{x^4+3x^3+x^2y} & x^2-3x-y \\
\quad -3x^3-x^2y-9x^2-6xy & \\
\quad \underline{-3x^3 \qquad\quad -9x^2-3xy} & \\
\qquad\quad -x^2y \qquad -3xy-y^2 & \\
\qquad\quad \underline{-x^2y \qquad -3xy-y^2} &
\end{array}$$

8.

$$\begin{array}{l|l}
8a^2-26ab+15b^2 & 4a-3b \\ \cline{2-2}
\underline{8a^2-\ \ 6ab} & 2a-5b \\
\quad -20ab+15b^2 & \\
\quad \underline{-20ab+15b^2} &
\end{array}$$

9.

$$\begin{array}{l|l}
x^3+6x^2+9x+4 & x+4 \\ \cline{2-2}
\underline{x^3+4x^2} & x^2+2x+1 \\
\quad 2x^2+9x & \\
\quad \underline{2x^2+8x} & \\
\qquad\quad x+4 & \\
\qquad\quad \underline{x+4} &
\end{array}$$

10.
$$\begin{array}{rl}
x^3-3x+2 & \underline{\left| x^2-2x+1\right.} \\
\underline{x^3-2x^2+x} & x+2 \\
2x^2-4x+2 & \\
\underline{2x^2-4x+2} &
\end{array}$$

11.
$$\begin{array}{rl}
x^3-5x^2-\ x+14 & \underline{\left| x^2-3x-7\right.} \\
\underline{x^3-3x^2-7x} & x-2 \\
-2x^2+6x+14 & \\
\underline{-2x^2+6x+14} &
\end{array}$$

12.
$$\begin{array}{rl}
a^4-\ a^2b^2-12b^4 & \underline{\left| a^2+3b^2\right.} \\
\underline{a^4+3a^2b^2} & a^2-4b^2 \\
-4a^2b^2-12b^4 & \\
\underline{-4a^2b^2-12b^4} &
\end{array}$$

13.
$$\begin{array}{rl}
x^3-86x-140 & \underline{\left| x-10\right.} \\
\underline{x^3-10x^2} & x^2+10x+14 \\
10x^2-\ 86x & \\
\underline{10x^2-100x} & \\
14x-140 & \\
\underline{14x-140} &
\end{array}$$

14.
$$\begin{array}{rl}
x^3-5x^2-46x-40 & \underline{\left| x+4\right.} \\
\underline{x^3+4x^2} & x^2-9x-10 \\
-9x^2-46x & \\
\underline{-9x^2-36x} & \\
-10x-40 & \\
\underline{-10x-40} &
\end{array}$$

15.
$$\begin{array}{rl}
7x^3-24x^2+58x-21 & \underline{\left| 7x-3\right.} \\
\underline{7x^3-3x^2} & x^2-3x+7 \\
-21x^2+58x & \\
\underline{-21x^2+\ 9x} & \\
49x-21 & \\
\underline{49x-21} &
\end{array}$$

16.
$$\begin{array}{rl}
x^4-2x^3y+2x^2y^2-xy^3 & \underline{\left| x-y\right.} \\
\underline{x^4-\ x^3y} & x^3-x^2y+xy^2 \\
-\ x^3y+2x^2y^2 & \\
\underline{-\ x^3y+\ x^2y^2} & \\
x^2y^2-xy^3 & \\
\underline{x^2y^2-xy^3} &
\end{array}$$

17.
$$\begin{array}{rl}
x^4-5x^3+11x^2-12x+6 & \underline{\left| x^2-3x+3\right.} \\
\underline{x^4-3x^3+\ 3x^2} & x^2-2x+2 \\
-2x^3+\ 8x^2-12x & \\
\underline{-2x^3+\ 6x^2-\ 6x} & \\
2x^2-\ 6x+6 & \\
\underline{2x^2-\ 6x+6} &
\end{array}$$

18. $x^4+64 \;\big|\; x^2+4x+8$

$$\begin{array}{l}
x^4+4x^3+8x^2 \qquad\qquad x^2-4x+8 \\
\hline
-4x^3-8x^2 \\
-4x^3-16x^2-32x \\
\hline
8x^2+32x+64 \\
8x^2+32x+64 \\
\hline
\end{array}$$

19. $27x^3+8y^3 \;\big|\; 3x+2y$

$$\begin{array}{l}
27x^3+18x^2y \qquad\qquad 9x^2-6xy+4y^2 \\
\hline
-18x^2y \\
-18x^2y-12xy^2 \\
\hline
12xy^2+8y^3 \\
12xy^2+8y^3 \\
\hline
\end{array}$$

20. $x^4-16 \;\big|\; x-2$

$$\begin{array}{l}
x^4-2x^3 \qquad\qquad x^3+2x^2+4x+8 \\
\hline
2x^3 \\
2x^3-4x^2 \\
\hline
4x^2 \\
4x^2-8x \\
\hline
8x-16 \\
8x-16 \\
\hline
\end{array}$$

21. $x^6-2x^3+1 \;\big|\; x^2-2x+1$

$$\begin{array}{l}
x^6-2x^5+x^4 \qquad\qquad x^4+2x^3+3x^2+2x+1 \\
\hline
2x^5-x^4-2x^3 \\
2x^5-4x^4+2x^3 \\
\hline
3x^4-4x^3 \\
3x^4-6x^3+3x^2 \\
\hline
2x^3-3x^2 \\
2x^3-4x^2+2x \\
\hline
x^2-2x+1 \\
x^2-2x+1 \\
\hline
\end{array}$$

22. $7a^6-48a^4b^2+b^6 \;\big|\; a^2-3ab+b^2$

$$\begin{array}{l}
7a^6-21a^5b+7a^4b^2 \qquad\qquad 7a^4+21a^3b+8a^2b^2+3ab^3+b^4 \\
\hline
21a^5b-55a^4b^2 \\
21a^5b-63a^4b^2+21a^3b^3 \\
\hline
8a^4b^2-21a^3b^3 \\
8a^4b^2-24a^3b^3+8a^2b^4 \\
\hline
3a^3b^3-8a^2b^4 \\
3a^3b^3-9a^2b^4+3ab^5 \\
\hline
a^2b^4-3ab^5+b^6 \\
a^2b^4-3ab^5+b^6 \\
\hline
\end{array}$$

23.

$$
\begin{array}{l|l}
a^4-a^2b^2+4ab^3-4b^4 & a^2-ab+2b^2 \\ \cline{2-2}
\underline{a^4-a^3b+2a^2b^2} & a^2+ab-2b^2 \\
\quad a^3b-3a^2b^2+4ab^3 & \\
\quad \underline{a^3b-\ \ a^2b^2+2ab^3} & \\
\qquad -2a^2b^2+2ab^3-4b^4 & \\
\qquad \underline{-2a^2b^2+2ab^3-4b^4} &
\end{array}
$$

24.

$$
\begin{array}{l|l}
6x^4-96 & 3x-6 \\ \cline{2-2}
\underline{6x^4-12x^3} & 2x^3+4x^2+8x+16 \\
\quad 12x^3 & \\
\quad \underline{12x^3-24x^2} & \\
\qquad 24x^2 & \\
\qquad \underline{24x^2-48x} & \\
\qquad\quad 48x-96 & \\
\qquad\quad \underline{48x-96} &
\end{array}
$$

25.

$$
\begin{array}{l|l}
x^6+10x-33 & x^2-2x+3 \\ \cline{2-2}
\underline{x^6-2x^5+3x^4} & x^4+2x^3+x^2-4x-11 \\
\quad 2x^5-3x^4 & \\
\quad \underline{2x^5-4x^4+6x^3} & \\
\qquad x^4-6x^3 & \\
\qquad \underline{x^4-2x^3+\ \ 3x^2} & \\
\qquad\quad -4x^3-\ \ 3x^2+10x & \\
\qquad\quad \underline{-4x^3+\ \ 8x^2-12x} & \\
\qquad\qquad -11x^2+22x-33 & \\
\qquad\qquad \underline{-11x^2+22x-33} &
\end{array}
$$

26.

$$
\begin{array}{l|l}
x^4-81y^4 & x-3y \\ \cline{2-2}
\underline{x^4-3x^3y} & x^3+3x^2y+9xy^2+27y^3 \\
\quad 3x^3y & \\
\quad \underline{3x^3y-9x^2y^2} & \\
\qquad 9x^2y^2 & \\
\qquad \underline{9x^2y^2-27xy^3} & \\
\qquad\quad 27xy^3-81y^4 & \\
\qquad\quad \underline{27xy^3-81y^4} &
\end{array}
$$

27.

$$\begin{array}{l|l}
x^4-6x^3y+9x^2y^2-4y^4 & x^2-3xy+2y^2 \\
\underline{x^4-3x^3y+2x^2y^2} & x^2-3xy-2y^2 \\
\quad -3x^3y+7x^2y^2 & \\
\quad \underline{-3x^3y+9x^2y^2-6xy^3} & \\
\qquad\quad -2x^2y^2+6xy^3-4y^4 & \\
\qquad\quad \underline{-2x^2y^2+6xy^3-4y^4} &
\end{array}$$

28.

$$\begin{array}{l|l}
6x^3-16x^2y+6xy^2+4y^3 & 3x^2-2xy-y^2 \\
\underline{6x^3-4x^2y-2xy^2} & 2x-4y \\
\quad -12x^2y+8xy^2+4y^3 & \\
\quad \underline{-12x^2y+8xy^2+4y^3} &
\end{array}$$

29.

$$\begin{array}{l|l}
x^6-6x+5 & x^2-2x+1 \\
\underline{x^6-2x^5+x^4} & x^4+2x^3+3x^2+4x+5 \\
\quad 2x^5-x^4 & \\
\quad \underline{2x^5-4x^4+2x^3} & \\
\qquad 3x^4-2x^3 & \\
\qquad \underline{3x^4-6x^3+3x^2} & \\
\qquad\quad 4x^3-3x^2-6x & \\
\qquad\quad \underline{4x^3-8x^2+4x} & \\
\qquad\qquad 5x^2-10x+5 & \\
\qquad\qquad \underline{5x^2-10x+5} &
\end{array}$$

30.

$$\begin{array}{l|l}
1-x-3x^2-x^5 & 1+2x+x^2 \\
\underline{1+2x+x^2} & 1-3x+2x^2-x^3 \\
\quad -3x-4x^2 & \\
\quad \underline{-3x-6x^2-3x^3} & \\
\qquad 2x^2+3x^3 & \\
\qquad \underline{2x^2+4x^3+2x^4} & \\
\qquad\quad -\ x^3-2x^4-x^5 & \\
\qquad\quad \underline{-\ x^3-2x^4-x^5} &
\end{array}$$

31.

$$\begin{array}{l|l}
a^2b^2-a^2-b^2+1 & ab+a+b+1 \\ \cline{2-2}
\underline{a^2b^2+a^2b+ab^2+ab} & ab-a-b+1 \\
\quad -a^2b-ab^2-ab-a^2 & \\
\quad \underline{-a^2b \quad\quad -ab-a^2-a} & \\
\quad\quad -ab^2 \quad\quad +a-b^2 & \\
\quad\quad \underline{-ab^2-ab \quad\quad -b^2-b} & \\
\quad\quad\quad ab \quad +a \quad +b+1 & \\
\quad\quad\quad \underline{ab \quad +a \quad +b+1} &
\end{array}$$

32.

$$\begin{array}{l|l}
a^2b^2+2ab^2c-a^2c^2+b^2c^2 & ab+ac+bc \\ \cline{2-2}
\underline{a^2b^2+a^2bc+ab^2c} & ab-ac+bc \\
\quad -a^2bc+ab^2c-a^2c^2 & \\
\quad \underline{-a^2bc \quad\quad -a^2c^2-abc^2} & \\
\quad\quad ab^2c \quad\quad +abc^2+b^2c^2 & \\
\quad\quad \underline{ab^2c \quad\quad +abc^2+b^2c^2} &
\end{array}$$

33.

$$\begin{array}{l|l}
12a^4-26a^3b-8a^2b^2+10ab^3-8b^4 & 3a^2-2ab+b^2 \\ \cline{2-2}
\underline{12a^4-8a^3b+4a^2b^2} & 4a^2-6ab-8b^2 \\
\quad -18a^3b-12a^2b^2+10ab^3 & \\
\quad \underline{-18a^3b+12a^2b^2-6ab^3} & \\
\quad\quad -24a^2b^2+16ab^3-8b^4 & \\
\quad\quad \underline{-24a^2b^2+16ab^3-8b^4} &
\end{array}$$

34.

$$\begin{array}{l|l}
3a^2+8ab+4b^2+10ac+8bc+3c^2 & a+2b+3c \\ \cline{2-2}
\underline{3a^2+6ab \quad\quad +\ 9ac} & 3a+2b+c \\
\quad 2ab+4b^2+\quad ac+8bc & \\
\quad \underline{2ab+4b^2 \quad\quad +6bc} & \\
\quad\quad\quad ac+2bc+3c^2 & \\
\quad\quad\quad \underline{ac+2bc+3c^2} &
\end{array}$$

35.

$$\begin{array}{l|l}
x^3+y^3+z^3-3xyz & x^2-xy-xz+y^2-yz+z^2 \\ \cline{2-2}
\underline{x^3-x^2y-x^2z+xy^2-xyz+xz^2} & x+y+z \\
\quad x^2y+x^2z-xy^2-xz^2-2xyz+y^3 & \\
\quad \underline{x^2y \quad -xy^2 \quad -\ xyz+y^3-y^2z+yz^2} & \\
\quad\quad x^2z \quad -xz^2-\ xyz \quad +y^2z-yz^2+z^3 & \\
\quad\quad \underline{x^2z \quad -xz^2-\ xyz \quad +y^2z-yz^2+z^3} &
\end{array}$$

36.

$$a^6-b^6 \quad \big| \; a^3-2a^2b+2ab^2-b^3$$
$$a^6-2a^5b+2a^4b^2-a^3b^3 \qquad a^3+2a^2b+2ab^2+b^3$$
$$2a^5b-2a^4b^2+a^3b^3$$
$$2a^5b-4a^4b^2+4a^3b^3-2a^2b^4$$
$$2a^4b^2-3a^3b^3+2a^2b^4$$
$$2a^4b^2-4a^3b^3+4a^2b^4-2ab^5$$
$$a^3b^3-2a^2b^4+2ab^5-b^6$$
$$a^3b^3-2a^2b^4+2ab^5-b^6$$

37.

$$1-5x+10x^2-10x^3+5x^4-x^5 \quad \big| \; 1-3x+3x^2-x^3$$
$$1-3x+3x^2-x^3 \qquad 1-2x+x^2$$
$$-2x+7x^2-9x^3+5x^4$$
$$-2x+6x^2-6x^3+2x^4$$
$$x^2-3x^3+3x^4-x^5$$
$$x^2-3x^3+3x^4-x^5$$

38.

$$1-81y^4 \quad \big| \; 1-3y$$
$$1-3y \qquad 1+3y+9y^2+27y^3$$
$$3y$$
$$3y-9y^2$$
$$9y^2$$
$$9y^2-27y^3$$
$$27y^3-81y^4$$
$$27y^3-81y^4$$

39.

$$a^5+31a-56 \quad \big| \; a^2+4a+7$$
$$a^5+4a^4+7a^3 \qquad a^3-4a^2+9a-8$$
$$-4a^4-7a^3$$
$$-4a^4-16a^3-28a^2$$
$$9a^3+28a^2+31a$$
$$9a^3+36a^2+63a$$
$$-8a^2-32a-56$$
$$-8a^2-32a-56$$

40.

$$\begin{array}{l|l} x^6-6x+5 & x^2-2x+1 \\ \cline{2-2} \underline{x^6-2x^5+x^4} & x^4+2x^3+3x^2+4x+5 \\ \quad 2x^5-x^4 & \\ \quad \underline{2x^5-4x^4+2x^3} & \\ \qquad 3x^4-2x^3 & \\ \qquad \underline{3x^4-6x^3+3x^2} & \\ \qquad\quad 4x^3-3x^2-6x & \\ \qquad\quad \underline{4x^3-8x^2+4x} & \\ \qquad\qquad 5x^2-10x+5 & \\ \qquad\qquad \underline{5x^2-10x+5} & \end{array}$$

41.

$$\begin{array}{l|l} x^6+2x^5-4x^4-2x^3+12x^2-2x-1 & x^2+2x-1 \\ \cline{2-2} \underline{x^6+2x^5-x^4} & x^4-3x^2+4x+1 \\ \quad -3x^4-2x^3+12x^2 & \\ \quad \underline{-3x^4-6x^3+3x^2} & \\ \qquad 4x^3+9x^2-2x & \\ \qquad \underline{4x^3+8x^2-4x} & \\ \qquad\quad x^2+2x-1 & \\ \qquad\quad \underline{x^2+2x-1} & \end{array}$$

42.

$$\begin{array}{l|l} x^8+2x^6+3x^4+2x^2+1 & x^4-2x^3+3x^2-2x+1 \\ \cline{2-2} \underline{x^8-2x^7+3x^6-2x^5+x^4} & x^4+2x^3+3x^2+2x+1 \\ \quad 2x^7-x^6+2x^5+2x^4 & \\ \quad \underline{2x^7-4x^6+6x^5-4x^4+2x^3} & \\ \qquad 3x^6-4x^5+6x^4-2x^3+2x^2 & \\ \qquad \underline{3x^6-6x^5+9x^4-6x^3+3x^2} & \\ \qquad\quad 2x^5-3x^4+4x^3-x^2 & \\ \qquad\quad \underline{2x^5-4x^4+6x^3-4x^2+2x} & \\ \qquad\qquad x^4-2x^3+3x^2-2x+1 & \\ \qquad\qquad \underline{x^4-2x^3+3x^2-2x+1} & \end{array}$$

43.

$$\begin{array}{l|l}
x^6-13x^4-12x+8 & x^2+3x-2 \\
\underline{x^6+3x^5-2x^4} & x^4-3x^3-2x^2-4 \\
\quad -3x^5-11x^4 & \\
\quad \underline{-3x^5-9x^4+6x^3} & \\
\qquad -2x^4-6x^3 & \\
\qquad \underline{-2x^4-6x^3+4x^2} & \\
\qquad\qquad -4x^2-12x+8 & \\
\qquad\qquad \underline{-4x^2-12x+8} &
\end{array}$$

44.

$$\begin{array}{l|l}
2x^6-9x^4-8x^3-1 & x^3+3x^2+3x+1 \\
\underline{2x^6+6x^5+6x^4+2x^3} & 2x^3-6x^2+3x-1 \\
\quad -6x^5-15x^4-10x^3 & \\
\quad \underline{-6x^5-18x^4-18x^3-6x^2} & \\
\qquad 3x^4+8x^3+6x^2 & \\
\qquad \underline{3x^4+9x^3+9x^2+3x} & \\
\qquad\qquad -x^3-3x^2-3x-1 & \\
\qquad\qquad \underline{-x^3-3x^2-3x-1} &
\end{array}$$

45.

$$\begin{array}{l|l}
3x^5-46x^3+43x^2-36 & x^2+5x+3 \\
\underline{3x^5+15x^4+9x^3} & 3x^3-15x^2+20x-12 \\
\quad -15x^4-55x^3+43x^2 & \\
\quad \underline{-15x^4-75x^3-45x^2} & \\
\qquad 20x^3+88x^2 & \\
\qquad \underline{20x^3+100x^2+60x} & \\
\qquad\qquad -12x^2-60x-36 & \\
\qquad\qquad \underline{-12x^2-60x-36} &
\end{array}$$

46.

$$\begin{array}{l|l}
a^{10}+16a^4-5a^2-15a-5 & a^4+2a^3-3a-1 \\
\underline{a^{10}+2a^9-3a^7-a^6} & a^6-2a^5+4a^4-5a^3+5a^2+5 \\
\quad -2a^9+3a^7+a^6 & \\
\quad \underline{-2a^9-4a^8+6a^6+2a^5} & \\
\qquad 4a^8+3a^7-5a^6-2a^5+16a^4 & \\
\qquad \underline{4a^8+8a^7-12a^5-4a^4} & \\
\qquad\quad -5a^7-5a^6+10a^5+20a^4 & \\
\qquad\quad \underline{-5a^7-10a^6+15a^4+5a^3} & \\
\qquad\qquad 5a^6+10a^5+5a^4-5a^3-5a^2 & \\
\qquad\qquad \underline{5a^6+10a^5-15a^3-5a^2} & \\
\qquad\qquad\qquad 5a^4+10a^3-15a-5 & \\
\qquad\qquad\qquad \underline{5a^4+10a^3-15a-5} &
\end{array}$$

47.
$$
\begin{array}{l|l}
x^{10}-29x^4+62x^3+3x^2+16x-352 & x^4+2x^3-5x-11 \\
\cline{2-2}
x^{10}+2x^9-5x^7-11x^6 & x^6-2x^5+4x^4-3x^3+7x^2-16x+32 \\
\quad -2x^9+5x^7+11x^6 & \\
\quad -2x^9-4x^8+10x^6+22x^5 & \\
\qquad 4x^8+5x^7+x^6-22x^5-29x^4 & \\
\qquad 4x^8+8x^7-20x^5-44x^4 & \\
\qquad\quad -3x^7+x^6-2x^5+15x^4+62x^3 & \\
\qquad\quad -3x^7-6x^6+15x^4+33x^3 & \\
\qquad\qquad 7x^6-2x^5+29x^3+3x^2 & \\
\qquad\qquad 7x^6+14x^5-35x^3-77x^2 & \\
\qquad\qquad\quad -16x^5+64x^3+80x^2+16x & \\
\qquad\qquad\quad -16x^5-32x^4+80x^2+176x & \\
\qquad\qquad\qquad 32x^4+64x^3-160x-352 & \\
\qquad\qquad\qquad 32x^4+64x^3-160x-352 &
\end{array}
$$

48.
$$
\begin{array}{l|l}
x^7+389x^2y^5-149xy^6+115y^7 & x^3+3x^2y-xy^2+y^3 \\
\cline{2-2}
x^7+3x^6y-x^5y^2+x^4y^3 & x^4-3x^3y+10x^2y^2-34xy^3+115y^4 \\
\quad -3x^6y+x^5y^2-x^4y^3 & \\
\quad -3x^6y-9x^5y^2+3x^4y^3-3x^3y^4 & \\
\qquad 10x^5y^2-4x^4y^3+3x^3y^4+389x^2y^5 & \\
\qquad 10x^5y^2+30x^4y^3-10x^3y^4+10x^2y^5 & \\
\qquad\quad -34x^4y^3+13x^3y^4+379x^2y^5-149xy^6 & \\
\qquad\quad -34x^4y^3-102x^3y^4+34x^2y^5-34xy^6 & \\
\qquad\qquad 115x^3y^4+345x^2y^5-115xy^6+115y^7 & \\
\qquad\qquad 115x^3y^4+345x^2y^5-115xy^6+115y^7 &
\end{array}
$$

49.
$$
\begin{array}{l|l}
32x^8+4x^3-2x^2-1 & 4x^3+4x^2+2x+1 \\
\cline{2-2}
32x^8+32x^7+16x^6+8x^5 & 8x^5-8x^4+4x^3-2x^2+2x-1 \\
\quad -32x^7-16x^6-8x^5 & \\
\quad -32x^7-32x^6-16x^5-8x^4 & \\
\qquad 16x^6+8x^5+8x^4+4x^3 & \\
\qquad 16x^6+16x^5+8x^4+4x^3 & \\
\qquad\quad -8x^5-2x^2 & \\
\qquad\quad -8x^5-8x^4-4x^3-2x^2 & \\
\qquad\qquad 8x^4+4x^3 & \\
\qquad\qquad 8x^4+8x^3+4x^2+2x & \\
\qquad\qquad\quad -4x^3-4x^2-2x-1 & \\
\qquad\qquad\quad -4x^3-4x^2-2x-1 &
\end{array}
$$

50.

$$
\begin{array}{l|l}
m^{10}-6m^3+5m-2 & m^4+2m^3-3m-2 \\ \cline{2-2}
\underline{m^{10}+2m^9-3m^7-2m^6} & m^6-2m^5+4m^4-5m^3+6m^2-4m+1 \\
\quad -2m^9+3m^7+2m^6 & \\
\quad \underline{-2m^9-4m^8+6m^6+4m^5} & \\
\qquad 4m^8+3m^7-4m^6-4m^5 & \\
\qquad \underline{4m^8+8m^7-12m^5-8m^4} & \\
\qquad\quad -5m^7-4m^6+8m^5+8m^4-6m^3 & \\
\qquad\quad \underline{-5m^7-10m^6+15m^4+10m^3} & \\
\qquad\qquad 6m^6+8m^5-7m^4-16m^3 & \\
\qquad\qquad \underline{6m^6+12m^5-18m^3-12m^2} & \\
\qquad\qquad\quad -4m^5-7m^4+2m^3+12m^2+5m & \\
\qquad\qquad\quad \underline{-4m^5-8m^4+12m^2+8m} & \\
\qquad\qquad\qquad m^4+2m^3-3m-2 & \\
\qquad\qquad\qquad \underline{m^4+2m^3-3m-2} &
\end{array}
$$

51.

$$
\begin{array}{l|l}
x^8-51x^5y^3-56x^2y^6+186xy^7-95y^8 & x^3-3x^2y-4xy^2+5y^3 \\ \cline{2-2}
\underline{x^8-3x^7y-4x^6y^2+5x^5y^3} & x^5+3x^4y+13x^3y^2-5x^2y^3+22xy^4 \\
\quad 3x^7y+4x^6y^2-56x^5y^3 & \qquad -19y^5 \\
\quad \underline{3x^7y-9x^6y^2-12x^5y^3+15x^4y^4} & \\
\qquad 13x^6y^2-44x^5y^3-15x^4y^4 & \\
\qquad \underline{13x^6y^2-39x^5y^3-52x^4y^4+65x^3y^5} & \\
\qquad\quad -5x^5y^3+37x^4y^4-65x^3y^5-56x^2y^6 & \\
\qquad\quad \underline{-5x^5y^3+15x^4y^4+20x^3y^5-25x^2y^6} & \\
\qquad\qquad 22x^4y^4-85x^3y^5-31x^2y^6+186xy^7 & \\
\qquad\qquad \underline{22x^4y^4-66x^3y^5-88x^2y^6+110xy^7} & \\
\qquad\qquad\quad -19x^3y^5+57x^2y^6+76xy^7-95y^8 & \\
\qquad\qquad\quad \underline{-19x^3y^5+57x^2y^6+76xy^7-95y^8} &
\end{array}
$$

52.
$$\begin{array}{l|l}
a^8-7a^7+49a^5+24a^3+52a^2-19a+20 & a^3-5a^2+3a-4 \\ \cline{2-2}
\underline{a^8-5a^7+3a^6-4a^5} & a^5-2a^4-13a^3-6a^2+a-5 \\
\quad -2a^7-3a^6+53a^5 & \\
\quad \underline{-2a^7+10a^6-6a^5+8a^4} & \\
\qquad -13a^6+59a^5-8a^4+24a^3 & \\
\qquad \underline{-13a^6+65a^5-39a^4+52a^3} & \\
\qquad\quad -6a^5+31a^4-28a^3+52a^2 & \\
\qquad\quad \underline{-6a^5+30a^4-18a^3+24a^2} & \\
\qquad\qquad a^4-10a^3+28a^2-19a & \\
\qquad\qquad \underline{a^4-5a^3+3a^2-4a} & \\
\qquad\qquad\quad -5a^3+25a^2-15a+20 & \\
\qquad\qquad\quad \underline{-5a^3+25a^2-15a+20} &
\end{array}$$

53.
$$\begin{array}{l|l}
m^6-57m^4n^2+58m^2n^4-9n^6 & m^3+7m^2n-4mn^2-3n^3 \\ \cline{2-2}
\underline{m^6+7m^5n-4m^4n^2-3m^3n^3} & m^3-7m^2n-4mn^2+3n^3 \\
\quad -7m^5n-53m^4n^2+3m^3n^3+58m^2n^4 & \\
\quad \underline{-7m^5n-49m^4n^2+28m^3n^3+21m^2n^4} & \\
\qquad -4m^4n^2-25m^3n^3+37m^2n^4 & \\
\qquad \underline{-4m^4n^2-28m^3n^3+16m^2n^4+12mn^5} & \\
\qquad\qquad 3m^3n^3+21m^2n^4-12mn^5-9n^6 & \\
\qquad\qquad \underline{3m^3n^3+21m^2n^4-12mn^5-9n^6} &
\end{array}$$

Art. 135.

1. $(mx+my)+(2x+2y) = m(x+y)+2(x+y) = (m+2)(x+y).$
   $(ax+bx)+(a+b) = x(a+b)+1(a+b) = (x+1)(a+b).$

2. $(4x+2ax)+(6+3a) = 2x(2+a)+3(2+a) = (2x+3)(2+a).$
   $(ac+bc)+(ad+bd) = c(a+b)+d(a+b) = (c+d)(a+b).$

3. $(ab+a)-(3b+3) = a(b+1)-3(b+1) = (a-3)(b+1).$
   $(3xy-y)-(6x-2) = y(3x-1)-2(3x-1) = (y-2)(3x-1).$

4. $(2x^2+3xy)-(2mx+3my) = x(2x+3y)-m(2x+3y) = (x-m)(2x+3y).$
   $(2x^2-6x)+(mx-3m) = 2x(x-3)+m(x-3) = (2x+m)(x-3).$

5. $(mq^2-qx+pq)-(mqx-x^2+px) = q(mq-x+p)-x(mq-x+p)$
   $= (q-x)(mq-x+p).$

6. $(ap+aq-as)-(cp+cq-cs) = a(p+q-s)-c(p+q-s)$
$= (a-c)(p+q-s).$

7. $(ar-ac+2an)+(2br-2bc+4bn) = a(r-c+2n)+2b(r-c+2n)$
$= (a+2b)(r-c+2n).$

8. $(ax-cx+2bx)+(ay-cy+2by) = x(a-c+2b)+y(a-c+2b)$
$= (x+y)(a-c+2b).$

9. $(3ab+ad)+(6bc+2cd)-(12abc+4acd) = a(3b+d)+2c(3b+d)$
$-4ac(3b+d) = (a+2c-4ac)(3b+d).$

10. $(2am-2ad)-(bm-bd)-(cm-cd) = 2a(m-d)-b(m-d)-c(m-d)$
$= (2a-b-c)(m-d).$

11. $(3an+2ar+4ac)-(6bn+4br+8bc) = a(3n+2r+4c)-2b(3n+2r+4c)$
$= (a-2b)(3n+2r+4c).$

12. $(an+2ar-acd)+(bns+2brs-bcds) = a(n+2r-cd)+bs(n+2r-cd)$
$= (a+bs)(n+2r-cd).$

13. $(2ax-ay-az)+(2cx-cy-cz) = a(2x-y-z)+c(2x-y-z)$
$= (a+c)(2x-y-z).$

14. $(6nx-2px+4sx)+(9ny-3py+6sy) = 2x(3n-p+2s)+3y(3n-p+2s)$
$= (2x+3y)(3n-p+2s).$

15. $(6ax+4bx-2x)-(6ay+4by-2y) = 2x(3a+2b-1)-2y(3a+2b-1)$
$= (2x-2y)(3a+2b-1).$

16. $(21x-14y-35)+(3nx-2ny-5n) = 7(3x-2y-5)+n(3x-2y-5)$
$= (7+n)(3x-2y-5).$

Art. 139.

1. $(x^2+2ax+a^2)+6x+6a = (x+a)(x+a)+6(x+a) = (x+a+6)(x+a).$

2. $(x^2-4ax+4a^2)+9x-18a = (x-2a)(x-2a)+9(x-2a)$
$= (x-2a+9)(x-2a).$

3. $(m^2-10m+25)-mn+5n = (m-5)(m-5)-n(m-5)$
$= (m-5-n)(m-5).$

4. $(x^2-14x+49)-5ax+35a = (x-7)(x-7)-5a(x-7)$
$= (x-7-5a)(x-7).$

5. $(a^2-6ax+9x^2)+2a-6x = (a-3x)(a-3x)+2(a-3x)$
$= (a-3x+2)(a-3x).$

6. $(y^2+2yz+z^2)+by^2+byz = (y+z)(y+z)+by(y+z) = (y+z+by)(y+z).$

7. $(b^3-3b^2c+3bc^2-c^3)-4ab+4ac = (b^2-2bc+c^2)(b-c)-4a(b-c)$
$= (b^2-2bc+c^2-4a)(b-c).$

8. $(25n^2-10mn+m^2)-30nr+6mr = (5n-m)(5n-m)-6r(5n-m)$
$= (5n-m-6r)(5n-m).$

9. $(9x^2-24xy+16y^2)-15xz+20yz = (3x-4y)(3x-4y)-5z(3x-4y)$
$= (3x-4y-5z)(3x-4y).$

10. $(49x^2-28x+4)+21xy-6y = (7x-2)(7x-2)+3y(7x-2)$
$= (7x-2+3y)(7x-2).$

11. $(m^4-6m^2n^2+9n^4)-6m^2p+18n^2p = (m^2-3n^2)(m^2-3n^2)-6p(m^2-3n^2)$
$= (m^2-3n^2-6p)(m^2-3n^2).$

12. $(4n^2+4n+1)-10mn-5m = (2n+1)(2n+1)-5m(2n+1)$
$= (2n+1-5m)(2n+1).$

13. $(x^2+6xy+9y^2)-2cx-6cy = (x+3y)(x+3y)-2c(x+3y)$
$= (x+3y-2c)(x+3y).$

14. $(x^2-14xy+49y^2)-12x+84y = (x-7y)(x-7y)-12(x-7y)$
$= (x-7y-12)(x-7y).$

15. $(9x^2-12xy+4y^2)-12x+8y = (3x-2y)(3x-2y)-4(3x-2y)$
$= (3x-2y-4)(3x-2y).$

16. $(a^2x^2-6abx+9b^2)+11acx-33bc = (ax-3b)(ax-3b)+11c(ax-3b)$
$= (ax-3b+11c)(ax-3b).$

17. $(a^4-4a^2b+4b^2)-5a^2c+10bc = (a^2-2b)(a^2-2b)-5c(a^2-2b)$
$= (a^2-2b-5c)(a^2-2b).$

18. $(4a^2-4ab+b^2)-12ac+6bc = (2a-b)(2a-b)-6c(2a-b)$
$= (2a-b-6c)(2a-b).$

19. $(p^2+4pq+4q^2)-12pr-24qr = (p+2q)(p+2q)-12r(p+2q)$
$= (p+2q-12r)(p+2q).$

20. $(4c^2-12cd+9d^2)-6abc+9abd = (2c-3d)(2c-3d)-3ab(2c-3d)$
$= (2c-3d-3ab)(2c-3d).$

21. $(49a^2+42ab+9b^2)-91a-39b = (7a+3b)(7a+3b)-13(7a+3b)$
$= (7a+3b-13)(7a+3b).$

22. $(25a^2-20ab+4b^2)-10a^2b+4ab^2 = (5a-2b)(5a-2b)-2ab(5a-2b)$
$= (5a-2b-2ab)(5a-2b).$

23. $(a^2m^2+6am+9)+5amx+15x = (am+3)(am+3) + 5x(am+3)$
$= (am+3+5x)(am+3).$

24. $(9x^2+12x+4)+18ax+12a = (3x+2)(3x+2)+6a(3x+2)$
$= (3x+2+6a)(3x+2).$

25. $(25x^2+30xy+9y^2)-20x-12y = (5x+3y)(5x+3y)-4(5x+3y)$
$= (5x+3y-4)(5x+3y).$

Art. 140.

1. $a^4-81 = (a^2+9)(a^2-9) = (a^2+9)(a+3)(a-3).$

2. $x^5+y^5 = (x+y)(x^4-x^3y+x^2y^2-xy^3+y^4).$

3. $x^8-1 = (x^4+1)(x^4-1) = (x^4+1)(x^2+1)(x^2-1)$
$= (x^4+1)(x^2+1)(x+1)(x-1).$

4. $m^5-n^5 = (m-n)(m^4+m^3n+m^2n^2+mn^3+n^4).$

5. $x^6-y^6 = (x^3+y^3)(x^3-y^3) = (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2).$

6. $a^3-125 = (a-5)(a^2+5a+25).$

7. $a^5-1 = (a-1)(a^4+a^3+a^2+a+1).$

8. $x^3+64 = (x+4)(x^2-4x+16).$

9. $a^{12}-b^{12} = (a^6+b^6)(a^6-b^6) = (a^6+b^6)(a^3+b^3)(a^3-b^3)$
$= (a^6+b^6)(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2).$

10. $x^5+32 = (x+2)(x^4-2x^3+4x^2-8x+16).$

11. $x^3+8n^3 = (x+2n)(x^2-2nx+4n^2).$

12. $a^3+216 = (a+6)(a^2-6a+36).$

13. $x^3+1000 = (x+10)(x^2-10x+100).$

14. $x^4-100 = (x^2+10)(x^2-10).$

15. $1+x^7 = (1+x)(1-x+x^2-x^3+x^4-x^5+x^6)$.

16. $x^{15}-y^{15} = (x-y)(x^{14}+x^{13}y+x^{12}y^2+x^{11}y^3+x^{10}y^4+x^9y^5+x^8y^6+x^7y^7+x^6y^8$
$+x^5y^9+x^4y^{10}+x^3y^{11}+x^2y^{12}+xy^{13}+y^{14})$.

17. $x^3+343 = (x+7)(x^2-7x+49)$.

18. $a^5+1 = (a+1)(a^4-a^3+a^2-a+1)$.

19. $x^4-y^4 = (x^2+y^2)(x^2-y^2) = (x^2+y^2)(x+y)(x-y)$.

20. $a^{12}-1 = (a^6+1)(a^6-1) = (a^6+1)(a^3+1)(a^3-1)$
$= (a^6+1)(a+1)(a^2-a+1)(a-1)(a^2+a+1)$.

21. $a^2x^2-b^2y^2 = (ax+by)(ax-by)$.

22. $(mn)^2-(pq)^2 = (mn+pq)(mn-pq)$.

23. $x^4-625 = (x^2+25)(x^2-25) = (x^2+25)(x+5)(x-5)$.

24. $b^6-64 = (b^3+8)(b^3-8) = (b+2)(b^2-2b+4)(b-2)(b^2+2b+4)$.

25. $x^5+243 = (x+3)(x^4-3x^3+9x^2-27x+81)$

26. $x^2+4xy+4y^2 = (x+2y)(x+2y)$.

27. $a^2-6ab+9b^2 = (a-3b)(a-3b)$.

28. $4a^2-12ax+9x^2 = (2a-3x)(2a-3x)$.

29. $16m^2-40m+25 = (4m-5)(4m-5)$.

30. $x^2-10x+9 = (x-1)(x-9)$.

31. $x^2-34x+225 = (x-9)(x-25)$.

32. $x^2+5x-14 = (x+7)(x-2)$.

33. $x^4-26x^2+25 = (x^2-25)(x^2-1) = (x+5)(x-5)(x+1)(x-1)$.

34. $a^2+6a+8 = (a+4)(a+2)$.

35. $m^4-29m^2+100 = (m^2-25)(m^2-4) = (m+5)(m-5)(m+2)(m-2)$.

36. $x^2-5x-14 = (x-7)(x+2)$.

37. $16a^2+50a-21 = (2a+7)(8a-3)$.

38. $3b^2-2b-16 = (3b-8)(b+2)$.

39. $3y^2-25y+28 = (3y-4)(y-7)$.

40. $2z^2-7z+6 = (2z-3)(z-2)$.

41. $5a^2+22a+21 = (5a+7)(a+3)$.

42. $16x^4-62x^2+21 = (8x^2-3)(2x^2-7).$

43. $10p^2-41p+39 = (2p-3)(5p-13).$

44. $x^2-16x+64 = (x-8)(x-8).$

45. $10m^2+19m-15 = (5m-3)(2m+5).$

46. $6x^2+x-2 = (3x+2)(2x-1).$

47. $2x^6+x^3-6 = (2x^3-3)(x^3+2).$

48. $4x^4+13x^2-12 = (x^2+4)(4x^2-3).$

49. $4n^2-20n+25 = (2n-5)(2n-5).$

50. $49x^2y^2+154xy+121 = (7xy+11)(7xy+11).$

51. $a^2-b^2+2bc-c^2 = a^2-(b^2-2bc+c^2) = a^2-(b-c)^2$
$= (a-b+c)(a+b-c).$

52. $a^2b-bx^2+a^2x-x^3 = (a^2b-bx^2)+(a^2x-x^3) = b(a^2-x^2)+x(a^2-x^2)$
$= (b+x)(a^2-x^2) = (b+x)(a+x)(a-x).$

53. $x^2-2xy+y^2-4z^2 = (x-y)^2-4z^2 = (x-y+2z)(x-y-2z).$

54. $9x^2+20yz-4y^2-25z^2 = 9x^2-(4y^2-20yz+25z^2)$
$= 9x^2-(2y-5z)^2 = (3x+2y-5z)(3x-2y+5z).$

55. $ax^2-ab^2+b^2x-x^3 = (ax^2-ab^2)-(x^3-b^2x) = a(x^2-b^2)-x(x^2-b^2)$
$= (a-x)(x^2-b^2) = (a-x)(x+b)(x-b).$

56. $(x^2-yz)^2-16z^2 = (x^2-yz+4z)(x^2-yz-4z).$

57. $(a^2-b^2)^2-(c^2-d^2)^2 = (a^2-b^2+c^2-d^2)(a^2-b^2-c^2+d^2).$

58. $ax-cy+cz+ay-cx-az = (ax+ay-az)-(cx+cy-cz)$
$= a(x+y-z)-c(x+y-z) = (a-c)(x+y-z).$

59. $9x^2-15xy-6x+5y+1 = (9x^2-6x+1)-(15xy-5y)$
$= (3x-1)(3x-1)-5y(3x-1) = (3x-1-5y)(3x-1).$

60. $mx-nx+2by+my-ny+2bx = (mx-nx+2bx)+(my-ny+2by)$
$= x(m-n+2b)+y(m-n+2b) = (x+y)(m-n+2b).$

61. $6xy-2mx+9ny+6nx+4x^2-3my$
$= (6xy-3my+9ny)+(4x^2-2mx+6nx)$
$3y(2x-m+3n)+2x(2x-m+3n) = (3y+2x)(2x-m+3n).$

62. $x^3+2x^2-19x-20 = (x-4)(x+5)(x+1)$.

63. $4y^2-6yz+28y-21z+49 = (4y^2+28y+49)-(6yz+21z)$
$= (2y+7)(2y+7)-3z(2y+7) = (2y+7-3z)(2y+7)$.

64. $x^3-10x^2+31x-30 = (x-2)(x-3)(x-5)$.

65. $x^3-9x^2-22x+120 = (x-3)(x-10)(x+4)$.

66. $2ay-a^2-y^2+b^2+c^2-2bc = (b^2-2bc+c^2)-(a^2-2ay+y^2)$
$= (b-c)^2-(a-y)^2 = (b-c+a-y)(b-c-a+y)$.

67. $x^3-4x^2-17x+60 = (x-3)(x+4)(x-5)$.

68. $16x^2-40xy-24x+30y+25y^2 = (16x^2-40xy+25y^2)-(24x-30y)$
$= (4x-5y)(4x-5y)-6(4x-5y) = (4x-5y-6)(4x-5y)$.

69. $x^3+6x^2-27x-140 = (x+4)(x+7)(x-5)$.

70. $a^3+a^2b+a^2c-abc-b^2c-bc^2 = (a^3+a^2b+a^2c)-(abc+b^2c+bc^2)$
$= a^2(a+b+c)-bc(a+b+c) = (a^2-bc)(a+b+c)$.

71. $ab^2-abx-axy+aby+b^2x+bxy-bx^2-x^2y$
$= (ab^2+aby)-(abx+axy)+(b^2x+bxy)-(bx^2+x^2y)$
$= ab(b+y)-ax(b+y)+bx(b+y)-x^2(b+y)$
$= (ab-ax+bx-x^2)(b+y) = [(ab-ax)+(bx-x^2)](b+y)$
$= [a(b-x)+x(b-x)](b+y) = (a+x)(b-x)(b+y)$.

72. $2mx-4mr+3nx-cx-6nr+2cr = (2mx+3nx-cx)-(4mr+6nr-2cr)$
$= x(2m+3n-c)-2r(2m+3n-c) = (x-2r)(2m+3n-c)$.

73. $a^2-b^2-c^2+d^2-2ad+2bc = (a^2-2ad+d^2)-(b^2-2bc+c^2)$
$= (a-d)^2-(b-c)^2 = (a-d+b-c)(a-d-b+c)$.

74. $a^2+b^2c^2+5bcm+2abc+5am = (a^2+2abc+b^2c^2)+(5am+5bcm)$
$= (a+bc)(a+bc)+5m(a+bc) = (a+bc+5m)(a+bc)$.

75. $a^3+4a^2m+am^2-6m^3 = (a+3m)(a+2m)(a-m)$.

76. $x^3+2x^2y-23xy^2-60y^3 = (x-5y)(x+4y)(x+3y)$.

77. $ax+2bx+cx-ay-2by-cy = (ax+2bx+cx)-(ay+2by+cy)$
$= x(a+2b+c)-y(a+2b+c) = (x-y)(a+2b+c)$.

Art. 144.

1.
$$\begin{aligned} 9ax^2y^2 &= 3\times 3\times a\times x^2\times y^2\\ 15a^2xz &= 3\times 5\times a^2\times x\times z\\ \hline \text{H. C. D.} &= 3\times a\times x = 3ax. \end{aligned}$$

2.
$$\begin{aligned} ab^2cu^2v &= a\times b^2\times c\times u^2\times v\\ 3a^2buv^2w &= 3\times a^2\times b\times u\times v^2\times w\\ \hline \text{H. C. D.} &= a\times b\times u\times v = abuv. \end{aligned}$$

3.
$$\begin{aligned} 4ab^2 &= 2\times 2\times a\times b^2\\ 12a^2bx &= 2\times 2\times 3\times a^2\times b\times x\\ 8ab^2y &= 2\times 2\times 2\times a\times b^2\times y\\ \hline \text{H. C. D.} &= 2\times 2\times a\times b = 4ab. \end{aligned}$$

4.
$$\begin{aligned} 81x^2y^2z &= 3\times 3\times 3\times 3\times x^2\times y^2\times z\\ 45abx^2y^4z^3 &= 3\times 3\times 5\times a\times b\times x^2\times y^4\times z^3\\ \hline \text{H. C. D.} &= 3\times 3\times x^2\times y^2\times z = 9x^2y^2z. \end{aligned}$$

5.
$$\begin{aligned} a^2+2ab+b^2 &= (a+b)(a+b)\\ a^2-b^2 &= (a+b)(a-b)\\ \hline \text{H. C. D.} &= a+b. \end{aligned}$$

6.
$$\begin{aligned} x^2-1 &= (x+1)(x-1)\\ x^2-x-2 &= (x+1)(x-2)\\ \hline \text{H. C. D.} &= x+1. \end{aligned}$$

7.
$$\begin{aligned} m^3+n^3 &= (m+n)(m^2-mn+n^2)\\ m^2-n^2 &= (m+n)(m-n)\\ \hline \text{H. C. D.} &= m+n. \end{aligned}$$

8.
$$\begin{aligned} a^2+2a-3 &= (a+3)(a-1)\\ a^2+5a+6 &= (a+3)(a+2)\\ \hline \text{H. C. D.} &= a+3. \end{aligned}$$

9.
$$\begin{aligned} x^2-3x-10 &= (x+2)(x-5)\\ x^2-8x+15 &= (x-3)(x-5)\\ \hline \text{H. C. D.} &= x-5. \end{aligned}$$

10.

$$\begin{aligned} 3x^3-3x^2-126x &= 3x(x^2-x-42) = 3x(x-7)(x+6) \\ 5x^2-55x+140 &= 5(x^2-11x+28) = 5(x-7)(x-4) \\ \text{H. C. D.} &= x-7. \end{aligned}$$

11.

$$\begin{aligned} m^2-m-42 &= (m-7)(m+6) \\ m^2+15m+54 &= (m+9)(m+6) \\ \text{H. C. D.} &= m+6. \end{aligned}$$

12.

$$\begin{aligned} b^2-b-110 &= (b+10)(b-11) \\ b^2-16b+55 &= (b-5)(b-11) \\ \text{H. C. D.} &= b-11. \end{aligned}$$

13.

$$\begin{aligned} 5a(x^2+6x+5) &= 5a(x+5)(x+1) \\ 5mx(x^2-x-30) &= 5mx(x+5)(x-6) \\ \text{H. C. D.} &= 5(x+5) = 5x+25. \end{aligned}$$

14.

$$\begin{aligned} x^2-13x+36 &= (x-4)(x-9) \\ x^2+x-90 &= (x+10)(x-9) \\ \text{H. C. D.} &= x-9. \end{aligned}$$

15.

$$\begin{aligned} x^2-23x+132 &= (x-11)(x-12) \\ x^2-4x-77 &= (x-11)(x+7) \\ \text{H. C. D.} &= x-11. \end{aligned}$$

16.

$$\begin{aligned} 2x^2-x-15 &= (2x+5)(x-3) \\ 6x^2+13x-5 &= (2x+5)(3x-1) \\ \text{H. C. D.} &= 2x+5. \end{aligned}$$

17.

$$\begin{aligned} 6x^2-19x+10 &= (3x-2)(2x-5) \\ 3x^2-14x+8 &= (3x-2)(x-4) \\ \text{H. C. D.} &= 3x-2. \end{aligned}$$

18.

$$\begin{aligned} x^3+27 &= (x+3)(x^2-3x+9) \\ x^2-9 &= (x+3)(x-3) \\ 2x^2+5x-3 &= (x+3)(2x-1) \\ \text{H. C. D.} &= x+3. \end{aligned}$$

19.

$$\begin{aligned} x^2+4x+4 &= (x+2)(x+2) \\ x^3+8 &= (x+2)(x^2-2x+4) \\ 2x^2+x-6 &= (x+2)(2x-3) \\ \text{H. C. D} &= x+2. \end{aligned}$$

20.
$$\begin{aligned} x^2+6x+9 &= (x+3)(x+3)\\ x^2+x-6 &= (x+3)(x-2)\\ 3x^2+7x-6 &= (x+3)(3x-2)\\ \hline \text{H. C. D.} &= x+3. \end{aligned}$$

21.
$$\begin{aligned} x^3-125 &= (x-5)(x^2+5x+25)\\ x^2-10x+25 &= (x-5)(x-5)\\ 2x^2-11x+5 &= (x-5)(2x-1)\\ \hline \text{H. C. D.} &= x-5. \end{aligned}$$

22.
$$\begin{aligned} 8y^3-216 &= 8(y^3-27) = 8(y-3)(y^2+3y+9)\\ 4y^2-24y+36 &= 4(y^2-6y+9) = 4(y-3)(y-3)\\ 10y^2-26y-12 &= 2(5y^2-13y-6) = 2(y-3)(5y+2)\\ \hline \text{H. C. D.} &= 2(y-3) = 2y-6. \end{aligned}$$

23.
$$\begin{aligned} a^3-27b^3 &= (a-3b)(a^2+3ab+9b^2)\\ a^2-6ab+9b^2 &= (a-3b)(a+3b)\\ 2a^2-ab-15b^2 &= (a-3b)(2a+5b)\\ \hline \text{H. C. D.} &= a-3b. \end{aligned}$$

24.
$$\begin{aligned} 3a^5-48a &= 3a(a^4-16) = 3a(a^2+4)(a^2-4) = 3a(a^2+4)(a-2)(a+2)\\ 2a^3b-16b &= 2b(a^3-8) = 2b(a-2)(a^2+2a+4)\\ 5a^2c-20c &= 5c(a^2-4) = 5c(a-2)(a+2)\\ \hline \text{H. C. D.} &= a-2. \end{aligned}$$

Art. 147.

1.
$$\begin{array}{rl} x^2+2x-3\ ) & x^2+5x+6\ (\ 1\\ & \underline{x^2+2x-3}\\ & 3\ )\ \underline{3x+9}\\ & \qquad x+3\ )\ x^2+2x-3\ (\ x-1\\ & \qquad\qquad \underline{x^2+3x}\\ & \qquad\qquad -x-3\\ & \qquad\qquad \underline{-x-3} \end{array}$$

Hence, H. C. D. $= x+3$.

2.

$$x^3+4x^2+x-6 \;)\; x^3-6x^2+11x-6 \;(\; 1$$
$$\underline{x^3+4x^2+x-6}$$
$$-10x \;)\; \underline{-10x^2+10x}$$
$$x-1 \;)\; x^3+4x^2+x-6 \;(\; x^2+5x+6$$
$$\underline{x^3-x^2}$$
$$5x^2+x-6$$
$$\underline{5x^2-5x}$$
$$6x-6$$
$$\underline{6x-6}$$

Hence, H. C. D. $= x-1$.

3.

$$3x^2-4x+1$$
$$\underline{2}$$
$$2x^2+x-3 \;)\; 6x^2-8x+2 \;(\; 3$$
$$\underline{6x^2+3x-9}$$
$$-11 \;)\; \underline{-11x+11}$$
$$x-1 \;)\; 2x^2+x-3 \;(\; 2x+3$$
$$\underline{2x^2-2x}$$
$$3x-3$$
$$\underline{3x-3}$$

Hence, H. C. D. $= x-1$.

4.

$$4 \;)\; \underline{12x^2-88x+128}$$
$$3x^2-22x+32 \;)\; 3x^3-33x^2+96x-84 \;(\; x+11$$
$$\underline{3x^3-22x^2+32x}$$
$$-11x^2+64x-84$$
$$\underline{-3}$$
$$33x^2-192x+252$$
$$\underline{33x^2-242x+352}$$
$$50 \;)\; \underline{50x-100}$$
$$x-2 \;)\; 3x^2-22x+32 \;(\; 3x-16$$
$$\underline{3x^2-6x}$$
$$-16x+32$$
$$\underline{-16x+32}$$

Hence, H. C. D. $= x-2$.

5.

$$\begin{array}{l}
2x^3+9x^2+4x-15\ )\ 4x^3+8x^2+3x+20\ (\ 2 \\
\qquad\qquad\qquad\qquad\ \ \underline{4x^3+18x^2+8x-30} \\
\qquad\qquad\qquad\quad -5\ )\ \underline{-10x^2-5x+50} \\
\qquad\qquad\qquad\qquad\qquad 2x^2+x-10\ )\ 2x^3+9x^2+4x-15\ (\ x+4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \underline{2x^3+x^2-10x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 8x^2+14x-15 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{8x^2+4x-40} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 5\ )\ \underline{10x+25} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 2x+5
\end{array}$$

$$\begin{array}{l}
2x+5\ )\ 2x^2+x-10\ (\ x-2 \\
\qquad\quad\ \ \underline{2x^2+5x} \\
\qquad\qquad\quad -4x-10 \\
\qquad\qquad\quad \underline{-4x-10}
\end{array}$$

Hence, H. C. D. $= 2x+5$.

6.

$$\begin{array}{l}
\qquad\qquad\qquad\qquad\qquad\quad x^4-9x^3+29x^2-39x+18 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4} \\
4x^3-27x^2+56x-33\ )\ 4x^4-36x^3+116x^2-156x+72\ (\ x+9 \\
\qquad\qquad\qquad\qquad\quad \underline{4x^4-27x^3+56x^2-33x} \\
\qquad\qquad\qquad\qquad\qquad -9x^3+60x^2-123x+72 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-4} \\
\qquad\qquad\qquad\qquad\qquad\ 36x^3-240x^2+492x-288 \\
\qquad\qquad\qquad\qquad\qquad\ \underline{36x^3-243x^2+504x-297} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 3\ )\ \underline{3x^2-12x+9} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x^2-4x+3
\end{array}$$

$$\begin{array}{l}
x^2-4x+3\ )\ 4x^3-27x^2+56x-33\ (\ 4x-11 \\
\qquad\qquad\quad\ \underline{4x^3-16x^2+12x} \\
\qquad\qquad\qquad -11x^2+44x-33 \\
\qquad\qquad\qquad \underline{-11x^2+44x-33}
\end{array}$$

Hence, H. C. D. $= x^2-4x+3$.

7.

$$25x^4+5x^3-x-1$$
$$4$$
$$20x^4+x^2-1\ )\ 100x^4+20x^3-4x-4\ (\ 5$$
$$100x^4+5x^2-5$$
$$20x^3-5x^2-4x+1\ )\ 20x^4+x^2-1\ (\ x$$
$$20x^4-5x^3-4x^2+x$$
$$5x^3+5x^2-x-1$$

$$5x^3+5x^2-x-1\ )\ 20x^3-5x^2-4x+1\ (\ 4$$
$$20x^3+20x^2-4x-4$$
$$-5\ )\ -25x^2+5$$
$$5x^2-1\ )\ 5x^3+5x^2-x-1\ (\ x+1$$
$$5x^3\quad -x$$
$$5x^2\quad -1$$
$$5x^2\quad -1$$

Hence, H. C. D. $=5x^2-1$.

8.

$$x^4-x^3-2x^2+3x-1\ )\ x^4-2x^3+4x^2-6x+3\ (\ 1$$
$$x^4-x^3-2x^2+3x-1$$
$$-x^3+6x^2-9x+4$$

$$-x^3+6x^2-9x+4\ )\ x^4-x^3-2x^2+3x-1\ (\ -x-5$$
$$x^4-6x^3+9x^2-4x$$
$$5x^3-11x^2+7x-1$$
$$5x^3-30x^2+45x-20$$
$$19\ )\ 19x^2-38x+19$$
$$x^2-2x+1\ )\ x^3-6x^2+9x-4\ (\ x-4$$
$$x^3-2x^2+x$$
$$-4x^2+8x-4$$
$$-4x^2+8x-4$$

Hence. H. C. D. $=x^2-2x+1$.

9. $$2x^6-96x^2+14 = 2(x^6-48x^2+7)$$

$$x^6-48x^2+7\ )\ 7x^6-\ 48x^4+\ 1\ (\ 7$$
$$\underline{7x^6-336x^2+49}$$
$$-48\ )\ \underline{-\ 48x^4+336x^2-48}$$
$$x^4-7x^2+1\ )\ x^6-48x^2+7\ (\ x^2+7$$
$$\underline{x^6-\ 7x^4+x^2}$$
$$7x^4-49x^2+7$$
$$\underline{7x^4-49x^2+7}$$

Hence, H. C. D. $= x^4-7x^2+1.$

10. $$x^6+x^4+x^3-x-2\ )\ 2x^6+x^5-x^3-x^2-1\ (\ 2$$
$$\underline{2x^6+2x^4+2x^3-2x-4}$$
$$x^5-2x^4-3x^3-x^2+2x+3$$

$$x^5-2x^4-3x^3-x^2+2x+3\ )\ x^6+x^4+x^3-x-2\quad (\ x+2$$
$$\underline{x^6-2x^5-3x^4-x^3+2x^2+3x}$$
$$2x^5+4x^4+2x^3-2x^2-4x-2$$
$$\underline{2x^5-4x^4-6x^3-2x^2+4x+6}$$
$$8\ )\ \underline{8x^4+8x^3-8x-8}$$
$$x^4+\ x^3-\ x-1$$

$$x^4+x^3-x-1\ )\ x^5-2x^4-3x^3-x^2+2x+3\ (\ x-3$$
$$\underline{x^5+\ x^4\qquad -x^2-\ x}$$
$$-3x^4-3x^3\qquad +3x+3$$
$$\underline{-3x^4-3x^3\qquad +3x+3}$$

Hence, H. C. D. $= x^4+x^3-x-1.$

11. $$x^6-6x+5\ )\ x^6+x^5-11x+9\ (\ 1$$
$$\underline{x^6\qquad -\ 6x+5}$$
$$x^5-\ 5x+4\ )\ x^6-6x+5\ (\ x$$
$$\underline{x^6-5x^2+\ 4x}$$
$$5\ )\ \underline{5x^2-10x+5}$$
$$x^2-\ 2x+1$$

$$x^2-2x+1\ )\ x^5-5x+4\quad (\ x^3+2x^2+3x+4$$
$$\underline{x^5-2x^4+x^3}$$
$$2x^4-x^3-5x+4$$
$$\underline{2x^4-4x^3+2x^2}$$
$$3x^3-2x^2-5x+4$$
$$\underline{3x^3-6x^2+3x}$$
$$4x^2-8x+4$$
$$\underline{4x^2-8x+4}$$

Hence, H. C. D. $= x^2-2x+1$.

12.

$$x^6+17x^2-13x+2\ )\ x^6-2x^4+25x^2-39x+14\ (\ 1$$
$$\underline{x^6\qquad +17x^2-13x+\ 2}$$
$$-2\ )\ \underline{-2x^4+8x^2-26x+12}$$
$$x^4-4x^2+13x-\ 6$$

$$x^4-4x^2+13x-6\ )\ x^6+17x^2-13x+2\quad (\ x^2+4$$
$$\underline{x^6-\ 4x^4+13x^3-\ 6x^2}$$
$$4x^4-13x^3+23x^2-13x+\ 2$$
$$\underline{4x^4\qquad -16x^2+52x-24}$$
$$-13\ )\ \underline{-13x^3+39x^2-65x+26}$$
$$x^3-\ 3x^2+\ 5x-\ 2\ )\ x^4-4x^2+13x-6\ (\ x+3$$
$$\underline{x^4-3x^3+5x^2-\ 2x}$$
$$3x^3-9x^2+15x-6$$
$$\underline{3x^3-9x^2+15x-6}$$

Hence, H. C. D. $= x^3-3x^2+5x-2$.

13.

$$x^5-17x^3+7x^2-9\ )\ x^5-16x^3+2x^2+5x-12\ (\ 1$$
$$\underline{x^5-17x^3+7x^2\qquad -\ 9}$$
$$x^3-5x^2+5x-\ 3$$

$$x^3-5x^2+5x-3\ )\ x^5-17x^3+7x^2-9\quad (\ x^2+5x+3$$
$$\underline{x^5-5x^4+\ 5x^3-\ 3x^2}$$
$$5x^4-22x^3+10x^2-9$$
$$\underline{5x^4-25x^3+25x^2-15x}$$
$$3x^3-15x^2+15x-9$$
$$\underline{3x^3-15x^2+15x-9}$$

Hence, H. C. D. $= x^3-5x^2+5x-3$.

14.
$$
\begin{array}{rl}
x^5-16x^3+2x^2+5x-12\ ) & 2x^5-5x^4-12x^3+4x^2-9 \quad (\ 2 \\
 & \underline{2x^5-32x^3+4x^2+10x-24} \\
-5\ ) & \underline{-5x^4+20x^3-10x+15} \\
 & x^4-4x^3+2x-3
\end{array}
$$

$$
\begin{array}{rl}
x^4-4x^3+2x-3\ ) & x^5-16x^3+2x^2+5x-12 \quad (\ x+4 \\
 & \underline{x^5-4x^4+2x^2-3x} \\
 & 4x^4-16x^3+8x-12 \\
 & \underline{4x^4-16x^3+8x-12}
\end{array}
$$

Hence, H. C. D. $= x^4-4x^3+2x-3$.

15. $a^2x^5+83a^2x^2-48a^2x+54a^2 = a^2(x^5+83x^2-48x+54)$.
$2mn^2x^5-48mn^2x^3-26mn^2x^2-36mn^2 = 2mn^2(x^5-24x^3-13x^2-18)$.

$$
\begin{array}{rl}
x^5+83x^2-48x+54\ ) & x^5-24x^3-13x^2-18 \quad (\ 1 \\
 & \underline{x^5+83x^2-48x+54} \\
-24\ ) & \underline{-24x^3-96x^2+48x-72} \\
 & x^3+4x^2-2x+3
\end{array}
$$

$$
\begin{array}{rl}
x^3+4x^2-2x+3\ ) & x^5+83x^2-48x+54 \quad (\ x^2-4x+18 \\
 & \underline{x^5+4x^4-2x^3+3x^2} \\
 & -4x^4+2x^3+80x^2-48x+54 \\
 & \underline{-4x^4-16x^3+8x^2-12x} \\
 & 18x^3+72x^2-36x+54 \\
 & \underline{18x^3+72x^2-36x+54}
\end{array}
$$

Hence, H. C. D. $= x^3+4x^2-2x+3$.

16. $a^5b-5a^3bc^2+9a^2bc^3+112abc^4+147bc^5$
$= b(a^5-5a^3c^2+9a^2c^3+112ac^4+147c^5)$.

$a^4x-5a^2c^2x-12ac^3x+28c^4x = x(a^4-5a^2c^2-12ac^3+28c^4)$.

$$
\begin{array}{rl}
a^4-5a^2c^2-12ac^3+28c^4\ ) & a^5-5a^3c^2+9a^2c^3+112ac^4+147c^5 \quad (\ a \\
 & \underline{a^5-5a^3c^2-12a^2c^3+28ac^4} \\
21c^3\ ) & \underline{21a^2c^3+84ac^4+147c^5} \\
 & a^2+4ac+7c^2
\end{array}
$$

$$a^2+4ac+7c^2\ )\ a^4-5a^2c^2-12ac^3+28c^4\ (\ a^2-4ac+4c^2$$
$$a^4+4a^3c+\ 7a^2c^2$$
$$-4a^3c-12a^2c^2-12ac^3+28c^4$$
$$-4a^3c-16a^2c^2-28ac^3$$
$$4a^2c^2+16ac^3+28c^4$$
$$4a^2c^2+16ac^3+28c^4$$

Hence, H. C. D. $= a^2+4ac+7c^2$.

17.

$$x^6+8x^3+9x^2-2\ )\ x^6-3x^4+3x^2-1\ (\ 1$$
$$x^6+8x^3+9x^2-2$$
$$-3x^4-8x^3-6x^2+1$$

(Divisor mult. by 3.)

$$-3x^4-8x^3-6x^2+1\ )\ 3x^6+24x^3+27x^2-6\quad (\ -x^2+4x-23$$
$$3x^6+8x^5+6x^4-x^2$$
$$2\ )\ -8x^5-6x^4+24x^3+28x^2-6$$
$$-4x^5-3x^4+12x^3+14x^2-3$$
$$3$$
$$-12x^5-\ 9x^4+\ 36x^3+42x^2-9$$
$$-12x^5-32x^4-\ 24x^3+\ 4x$$
$$23x^4+\ 60x^3+\ 42x^2-\ 4x-\ 9$$
$$3$$
$$69x^4+180x^3+126x^2-12x-27$$
$$69x^4+184x^3+138x^2\qquad -23$$
$$-4\ )\ -4x^3-\ 12x^2-12x-\ 4$$
$$x^3+\ \ 3x^2+\ 3x+\ 1$$

$$x^3+3x^2+3x+1\ )\ 3x^4+8x^3+6x^2-1\quad (\ 3x-1$$
$$3x^4+9x^3+9x^2+3x$$
$$-\ x^3-3x^2-3x-1$$
$$-\ x^3-3x^2-3x-1$$

Hence, H. C. D. $= x^3+3x^2+3x+1$.

18.

$$
\begin{array}{r}
x^5-17x^3+7x^2-9 \;)\; 3x^5-46x^3+43x^2-36 \;(\; 3 \\
\underline{3x^5-51x^3+21x^2-27} \\
5x^3+22x^2-9
\end{array}
$$

(Divisor mult. by 5.)

$$
\begin{array}{r}
5x^3+22x^2-9 \;)\; 5x^5-85x^3+35x^2-45 \;(\; x^2-22x+59 \\
\underline{5x^5+22x^4-\;9x^2} \\
-22x^4-85^3+44x^2-45 \\
\underline{5} \\
-110x^4-425x^3+220x^2-225 \\
\underline{-110x^4-484x^3+198x} \\
59x^3+220x^2-198x-225 \\
\underline{5} \\
295x^3+1100x^2-990x-1125 \\
\underline{295x^3-1298x^2 \qquad -\;531} \\
-198 \;)\; \underline{-\;198x^2-990x-\;594} \\
x^2+\;\;5x+\;\;\;3
\end{array}
$$

$$
\begin{array}{r}
x^2+5x+3 \;)\; 5x^3+22x^2-9 \;(\; 5x-3 \\
\underline{5x^3+25x^2+15x} \\
-\;3x^2-15x-9 \\
\underline{-\;3x^2-15x-9}
\end{array}
$$

Hence, H. C. D. $= x^2+5x+3$.

19.

$$
\begin{array}{r}
x^6-30x^3+32x^2-3 \;)\; x^6-8x^4+8x^2-1 \;(\; 1 \\
\underline{x^6-30x^3+32x^2-3} \\
-2 \;)\; \underline{-8x^4+30x^3-24x^2+2} \\
4x^4-15x^3+12x^2-1
\end{array}
$$

(Divisor mult. by 4.)

$$
\begin{array}{r}
4x^4-15x^3+12x^2-1 \;)\; 4x^6-120x^3+128x^2-12 \;(\; x^2+5x+59 \\
\underline{4x^6-15x^5+12x^4-x^2} \\
3 \;)\; \underline{15x^5-12x^4-120x^3+129x^2-12} \\
5x^5-\;4x^4-\;40x^3+\;43x^2-\;4
\end{array}
$$

$$5x^5-4x^4-40x^3+43x^2-4$$
$$4$$
$$20x^5-16x^4-160x^3+172x^2-16$$
$$20x^5-75x^4+\;60x^3-\;\;5x$$
$$59x^4-220x^3+172x^2+5x-16$$
$$4$$
$$236x^4-880x^3+688x^2+20x-64$$
$$236x^4-885x^3+708x^2\qquad-59$$
$$5\;)\;5x^3-\;20x^2+20x-\;5$$
$$x^3-\;\;4x^2+\;4x-\;1$$

$$x^3-4x^2+4x-1\;)\;4x^4-15x^3+12x^2-1\qquad(\;4x+1$$
$$4x^4-16x^3+16x^2-4x$$
$$x^3-\;4x^2+4x-1$$
$$x^3-\;4x^2+4x-1$$

Hence, H. C. D. $= x^3-4x^2+4x-1.$

20. $$x^5-27x^2-77x-245\;)\;x^5-5x^3+\;9x^2+112x+147\;(\;1$$
$$x^5\qquad-27x^2-\;77x-245$$
$$-5x^3+36x^2+189x+392$$

(Divisor mult. by 5.)

$$-5x^3+36x^2+189x+392\;)\;5x^5-135x^2-385x-1225\;(\;-x^2-36x-2241$$
$$5x^5-36x^4-189x^3-392x^2$$
$$36x^4+189x^3+257x^2-385x-1225$$
$$5$$
$$180x^4+\;945x^3+1285x^2-\;1925x-6125$$
$$180x^4-1296x^3-6804x^2-14112x$$
$$2241x^3+8089x^2+12187x-6125$$
$$5$$
$$11205x^3+40445x^2+\;60935x-\;30625$$
$$11205x^3-80676x^2-423549x-878472$$
$$121121\;)\;121121x^2+484484x+847847$$
$$x^2+\qquad4x+\qquad7$$

$$x^2+4x+7\ )\ -5x^3+36x^2+189x+392\ (\ -5x+56$$
$$-5x^3-20x^2-\ 35x$$
$$56x^2+224x+392$$
$$56x^2+224x+392$$

Hence, H. C. D. $= x^2+4x+7$.

21. $2x^5-10x^3-38x^2-98 = 2(x^5-5x^3-19x^2-49)$.

$3x^5+36x^2+237x+84 = 3(x^5+12x^2+79x+28)$.

$$x^5-5x^3-19x^2-49\ )\ x^5+12x^2+79x+28\ (\ 1$$
$$x^5-\ 5x^3-19x^2-49$$
$$5x^3+31x^2+79x+77$$

(Divisor mult. by 5.)

$$5x^3+31x^2+79x+77\ )\ 5x^5-25x^3-95x^2-245\ (\ x^2-31x+441$$
$$5x^5+31x^4+79x^3+77x^2$$
$$-31x^4-104x^3-172x^2-245$$
$$5$$
$$-155x^4-520x^3-\ 860x^2-1225$$
$$-155x^4-961x^3-2449x^2-2387x$$
$$441x^3+1589x^2+2387x-1225$$
$$5$$
$$2205x^3+\ 7945x^2+11935x-\ 6125$$
$$2205x^3+13671x^2+34839x+33957$$
$$-5726\ )\ -5726x^2-22904x-40082$$
$$x^2+\quad 4x+\quad 7$$

$$x^2+4x+7\ )\ 5x^3+31x^2+79x+77\ (\ 5x+11$$
$$5x^3+20x^2+35x$$
$$11x^2+44x+77$$
$$11x^2+44x+77$$

Hence, H. C. D. $= x^2+4x+7$.

22. $2abx^5-164abx-6ab = 2ab\,(x^5-82x-3).$

$$3m^2nx^5+84m^2nx^2-27m^2n = 3m^2n\,(x^5+28x^2-9).$$

$$
\begin{array}{r}
x^5-82x-3\;)\;x^5+28x^2-9\;(\;1\\
x^5-82x-3\\
\hline
2\;)\;28x^2+82x-6\\
\hline
14x^2+41x-3
\end{array}
$$

(Divisor mult. by 14.)

$$
\begin{array}{r}
14x^2+41x-3\;)\;14x^5-1148x-42\;(\;x^3-41x^2+1723x-72365\\
14x^5+41x^4-3x^3\\
\hline
-41x^4+3x^3-1148x-42\\
14\\
\hline
-574x^4+\;\;42x^3-16072x-588\\
-574x^4-1681x^3+\;\;123x^2\\
\hline
1723x^3-\;\;123x^2-16072x-588\\
14\\
\hline
24122x^3-\;\;1722x^2-225008x-8232\\
24122x^3+70643x^2-\;\;5169x\\
\hline
-72365x^2-219839x-8232\\
14\\
\hline
-1013110x^2-3077746x-115248\\
-1013110x^2-2966965x+217095\\
\hline
-110781\;)\;-110781x-332343\\
\hline
x+\;\;\;3
\end{array}
$$

$$
\begin{array}{r}
x+3\;)\;14x^2+41x-3\;(\;14x-1\\
14x^2+42x\\
\hline
-\;\;x-3\\
-\;\;x-3\\
\hline
\end{array}
$$

Hence, H. C. D. $= x+3$.

Art. 151.

1.
$$\begin{aligned} 4a^2b^3 &= 4\times a^2\times b^3 \\ 3a^4b &= 3\times a^4\times b \\ \hline \text{L. C. M.} &= 3\times 4\times a^4\times b^3 = 12a^4b^3. \end{aligned}$$

2.
$$\begin{aligned} 9a^3b^7 &= 3^2\times a^3\times b^7 \\ 15a^4b^6 &= 3\times 5\times a^4\times b^6 \\ \hline \text{L. C. M.} &= 3^2\times 5\times a^4\times b^7 = 45a^4b^7. \end{aligned}$$

3
$$\begin{aligned} 15x^3y^2z &= 3\times 5\times x^3\times y^2\times z \\ 21x^2y^3z^2 &= 3\times 7\times x^2\times y^3\times z^2 \\ \hline \text{L. C. M.} &= 3\times 5\times 7\times x^3\times y^3\times z^2 = 105x^3y^3z^2. \end{aligned}$$

4.
$$\begin{aligned} 6x^3yz^2 &= 2\times 3\times x^3\times y\times z^2 \\ 14x^2y^3z &= 2\times 7\times x^2\times y^3\times z \\ \hline \text{L. C. M.} &= 2\times 3\times 7\times x^3\times y^3\times z^2 = 42x^3y^3z^2. \end{aligned}$$

5.
$$\begin{aligned} 2x+1 &= 2x+1 \\ 2(4x^2-1) &= 2(2x+1)(2x-1) \\ \hline \text{L. C. M.} &= 2(2x+1)(2x-1) = 2(4x^2-1). \end{aligned}$$

6.
$$\begin{aligned} x^3-8 &= (x-2)(x^2+2x+4) \\ x^2-4 &= (x-2)(x+2) \\ \hline \text{L. C. M.} &= (x-2)(x+2)(x^2+2x+4) = x^4+2x^3-8x-16. \end{aligned}$$

7.
$$\begin{aligned} 6(x^2y-xy^2) &= 2\times 3\times xy(x-y) \\ 16(x^3-y^3) &= 2^4(x-y)(x^2+xy+y^2) \\ \hline \text{L. C. M.} &= 2^4\times 3\times xy(x-y)(x^2+xy+y^2) = 48xy(x^3-y^3). \end{aligned}$$

8.
$$\begin{aligned} x^2+2x-35 &= (x+7)(x-5) \\ x^2+x-42 &= (x+7)(x-6) \\ \hline \text{L. C. M.} &= (x+7)(x-5)(x-6) = x^3-4x^2-47x+210. \end{aligned}$$

9.
$$\begin{aligned} x^2-3x-4 &= (x-4)(x+1) \\ x^2-x-12 &= (x-4)(x+3) \\ \hline \text{L. C. M.} &= (x-4)(x+1)(x+3) = x^3-13x-12. \end{aligned}$$

10.
$$\begin{aligned} x^2+2x-99 &= (x+11)(x-9) \\ x^2-x-132 &= (x+11)(x-12) \\ \hline \text{L. C. M.} &= (x+11)(x-9)(x-12) = x^3-10x^2-123x+1188. \end{aligned}$$

11.
$$\begin{aligned} x^2-3x-28 &= (x+4)(x-7) \\ x^2+x-12 &= (x+4)(x-3) \\ x^2-10x+21 &= (x-7)(x-3) \\ \hline \text{L. C. M.} &= (x+4)(x-7)(x-3) = x^3-6x^2-19x+84. \end{aligned}$$

12.
$$\begin{aligned} \text{H. C. D.} &= x-1 \\ x^3-x^2+x-1 &= (x-1)(x^2+1) \\ x^3-7x+6 &= (x-1)(x^2+x-6) = (x-1)(x+3)(x-2) \\ \hline \text{L. C. M.} &= (x-1)(x^2+1)(x+3)(x-2) = x^5-6x^3+6x^2-7x+6. \end{aligned}$$

13.
$$\begin{aligned} 12x^2-x-1 &= (4x+1)(3x-1) \\ 6x^2-5x+1 &= (3x-1)(2x-1) \\ \hline \text{L. C. M.} &= (4x+1)(3x-1)(2x-1) = 24x^3-14x^2-x+1. \end{aligned}$$

14.
$$\begin{aligned} 3x^2-5x+2 &= (3x-2)(x-1) \\ 4x^3-4x^2-x+1 &= (x-1)(4x^2-1) = (x-1)(2x-1)(2x+1) \\ \hline \text{L. C. M.} &= (3x-2)(x-1)(2x-1)(2x+1) \\ &= 12x^4-20x^3+5x^2+5x-2. \end{aligned}$$

15.
$$\begin{aligned} \text{H. C. D.} &= a^2-a+7 \\ a^3+6a+7 &= (a^2-a+7)(a+1) \\ a^3-2a^2+8a-7 &= (a^2-a+7)(a-1) \\ \hline \text{L. C. M.} &= (a^2-a+7)(a+1)(a-1) = a^4-a^3+6a^2+a-7. \end{aligned}$$

16.
$$\begin{aligned} 15(a^2b-ab^2) &= 3\times5\times ab(a-b) \\ 21(a^3-ab^2) &= 3\times7\times a(a^2-b^2) = 3\times7\times a(a-b)(a+b) \\ 35(ab^2+b^3) &= 5\times7\times b^2(a+b) \\ \hline \text{L. C. M.} &= 3\times5\times7\times ab^2(a-b)(a+b) = 105ab^2(a^2-b^2). \end{aligned}$$

17.
$$\begin{aligned} x^2-1 &= (x-1)(x+1) \\ x^2+1 &= x^2+1 \\ x^4+1 &= x^4+1. \\ x^8-1 &= (x^4+1)(x^4-1) = (x^4+1)(x^2+1)(x^2-1) \\ &= (x^4+1)(x^2+1)(x+1)(x-1) \\ \hline \text{L. C. M.} &= (x^4+1)(x^2+1)(x+1)(x-1) = x^8-1. \end{aligned}$$

18. $$\text{H. C. D.} = x^2+2x-3$$
$$x^3+3x^2-x-3 = (x^2+2x-3)(x+1) = (x+3)(x-1)(x+1)$$
$$x^3+4x^2+x-6 = (x^2+2x-3)(x+2) = (x+3)(x-1)(x+2)$$
$$\text{L. C. M.} = (x+3)(x-1)(x+1)(x+2) = x^4+5x^3+5x^2-5x-6.$$

19. $$\text{H. C. D.} = x^2-2x-3$$
$$x^3-7x-6 = (x^2-2x-3)(x+2) = (x-3)(x+1)(x+2)$$
$$x^3+x^2-9x-9 = (x^2-2x-3)(x+3) = (x-3)(x+1)(x+3)$$
$$\text{L. C. M.} = (x-3)(x+1)(x+2)(x+3) = x^4+3x^3-7x^2-27x-18.$$

20. $$\text{H. C. D.} = x^3-8x^2+x+42$$
$$x^4-7x^3-7x^2+43x+42 = (x^3-8x^2+x+42)(x+1)$$
$$x^4-9x^3+9x^2+41x-42 = (x^3-8x^2+x+42)(x-1)$$
$$\text{L. C. M.} = (x^3-8x^2+x+42)(x+1)(x-1)$$
$$= x^5-8x^4+50x^3-x-42.$$

21. $$\text{H. C. D.} = x-3.$$
$$x^3-6x^2+11x-6 = (x-3)(x^2-3x+2) = (x-3)(x-2)(x-1)$$
$$x^3-9x^2+26x-24 = (x-3)(x^2-6x+8) = (x-3)(x-2)(x-4)$$
$$x^3-8x^2+19x-12 = (x-3)(x^2-5x+4) = (x-3)(x-1)(x-4)$$
$$\text{L. C. M.} = (x-3)(x-2)(x-1)(x-4)$$
$$= x^4-10x^3+35x^2-50x+24.$$

Art. 158.

1. $\dfrac{3ay}{y}+\dfrac{x}{y}=\dfrac{3ay+x}{y}.$

2. $\dfrac{2x}{x}+\dfrac{a}{x}=\dfrac{2x+a}{x}.$

3. $\dfrac{3ab}{b}+\dfrac{ab-a}{b}=\dfrac{3ab+(ab-a)}{b}=\dfrac{4ab-a}{b}.$

4. $\dfrac{3a^2x^2}{x}-\dfrac{a^2x^2-a^3}{x}=\dfrac{3a^2x^2-(a^2x^2-a^3)}{x}=\dfrac{2a^2x^2+a^3}{x}.$

5. $\dfrac{1+x}{1+x}+\dfrac{x}{1+x}=\dfrac{(1+x)+x}{1+x}=\dfrac{1+2x}{1+x}.$

6. $\dfrac{2a(a-b)}{a-b}+\dfrac{4b^2}{a-b}=\dfrac{2a^2-2ab+4b^2}{a-b}.$

10. $$\begin{aligned} x^2+2x-99 &= (x+11)(x-9) \\ x^2-x-132 &= (x+11)(x-12) \\ \hline \text{L. C. M.} &= (x+11)(x-9)(x-12) = x^3-10x^2-123x+1188. \end{aligned}$$

11. $$\begin{aligned} x^2-3x-28 &= (x+4)(x-7) \\ x^2+x-12 &= (x+4)(x-3) \\ x^2-10x+21 &= (x-7)(x-3) \\ \hline \text{L. C. M.} &= (x+4)(x-7)(x-3) = x^3-6x^2-19x+84. \end{aligned}$$

12. $$\begin{aligned} \text{H. C. D.} &= x-1 \\ x^3-x^2+x-1 &= (x-1)(x^2+1) \\ x^3-7x+6 &= (x-1)(x^2+x-6) = (x-1)(x+3)(x-2) \\ \hline \text{L. C. M.} &= (x-1)(x^2+1)(x+3)(x-2) = x^5-6x^3+6x^2-7x+6. \end{aligned}$$

13. $$\begin{aligned} 12x^2-x-1 &= (4x+1)(3x-1) \\ 6x^2-5x+1 &= (3x-1)(2x-1) \\ \hline \text{L. C. M.} &= (4x+1)(3x-1)(2x-1) = 24x^3-14x^2-x+1. \end{aligned}$$

14. $$\begin{aligned} 3x^2-5x+2 &= (3x-2)(x-1) \\ 4x^3-4x^2-x+1 &= (x-1)(4x^2-1) = (x-1)(2x-1)(2x+1) \\ \hline \text{L. C. M.} &= (3x-2)(x-1)(2x-1)(2x+1) \\ &= 12x^4-20x^3+5x^2+5x-2. \end{aligned}$$

15. $$\begin{aligned} \text{H. C. D.} &= a^2-a+7 \\ a^3+6a+7 &= (a^2-a+7)(a+1) \\ a^3-2a^2+8a-7 &= (a^2-a+7)(a-1) \\ \hline \text{L. C. M.} &= (a^2-a+7)(a+1)(a-1) = a^4-a^3+6a^2+a-7. \end{aligned}$$

16. $$\begin{aligned} 15(a^2b-ab^2) &= 3\times5\times ab(a-b) \\ 21(a^3-ab^2) &= 3\times7\times a(a^2-b^2) = 3\times7\times a(a-b)(a+b) \\ 35(ab^2+b^3) &= 5\times7\times b^2(a+b) \\ \hline \text{L. C. M.} &= 3\times5\times7\times ab^2(a-b)(a+b) = 105ab^2(a^2-b^2). \end{aligned}$$

17. $$\begin{aligned} x^2-1 &= (x-1)(x+1) \\ x^2+1 &= x^2+1 \\ x^4+1 &= x^4+1. \\ x^8-1 &= (x^4+1)(x^4-1) = (x^4+1)(x^2+1)(x^2-1) \\ &= (x^4+1)(x^2+1)(x+1)(x-1) \\ \hline \text{L. C. M.} &= (x^4+1)(x^2+1)(x+1)(x-1) = x^8-1. \end{aligned}$$

18.

$$\begin{aligned}
\text{H. C. D.} &= x^2+2x-3\\
x^3+3x^2-x-3 &= (x^2+2x-3)(x+1) = (x+3)(x-1)(x+1)\\
x^3+4x^2+x-6 &= (x^2+2x-3)(x+2) = (x+3)(x-1)(x+2)\\
\hline
\text{L. C. M.} &= (x+3)(x-1)(x+1)(x+2) = x^4+5x^3+5x^2-5x-6.
\end{aligned}$$

19.

$$\begin{aligned}
\text{H. C. D.} &= x^2-2x-3\\
x^3-7x-6 &= (x^2-2x-3)(x+2) = (x-3)(x+1)(x+2)\\
x^3+x^2-9x-9 &= (x^2-2x-3)(x+3) = (x-3)(x+1)(x+3)\\
\hline
\text{L. C. M.} &= (x-3)(x+1)(x+2)(x+3) = x^4+3x^3-7x^2-27x-18.
\end{aligned}$$

20.

$$\begin{aligned}
\text{H. C. D.} &= x^3-8x^2+x+42\\
x^4-7x^3-7x^2+43x+42 &= (x^3-8x^2+x+42)(x+1)\\
x^4-9x^3+9x^2+41x-42 &= (x^3-8x^2+x+42)(x-1)\\
\hline
\text{L. C. M.} &= (x^3-8x^2+x+42)(x+1)(x-1)\\
&= x^5-8x^4+50x^3-x-42.
\end{aligned}$$

21.

$$\begin{aligned}
\text{H. C. D.} &= x-3.\\
x^3-6x^2+11x-6 &= (x-3)(x^2-3x+2) = (x-3)(x-2)(x-1)\\
x^3-9x^2+26x-24 &= (x-3)(x^2-6x+8) = (x-3)(x-2)(x-4)\\
x^3-8x^2+19x-12 &= (x-3)(x^2-5x+4) = (x-3)(x-1)(x-4)\\
\hline
\text{L. C. M.} &= (x-3)(x-2)(x-1)(x-4)\\
&= x^4-10x^3+35x^2-50x+24.
\end{aligned}$$

Art. 158.

1. $\dfrac{3ay}{y}+\dfrac{x}{y}=\dfrac{3ay+x}{y}$.

2. $\dfrac{2x}{x}+\dfrac{a}{x}=\dfrac{2x+a}{x}$.

3. $\dfrac{3ab}{b}+\dfrac{ab-a}{b}=\dfrac{3ab+(ab-a)}{b}=\dfrac{4ab-a}{b}$.

4. $\dfrac{3a^2x^2}{x}-\dfrac{a^2x^2-a^3}{x}=\dfrac{3a^2x^2-(a^2x^2-a^3)}{x}=\dfrac{2a^2x^2+a^3}{x}$.

5. $\dfrac{1+x}{1+x}+\dfrac{x}{1+x}=\dfrac{(1+x)+x}{1+x}=\dfrac{1+2x}{1+x}$.

6. $\dfrac{2a(a-b)}{a-b}+\dfrac{4b^2}{a-b}=\dfrac{2a^2-2ab+4b^2}{a-b}$.

7. $$\frac{(a+x)(a-x)}{a-x}+\frac{a^2+x^2}{a-x}=\frac{(a^2-x^2)+a^2+x^2}{a-x}=\frac{2a^2}{a-x}.$$

8. $$\frac{3(x^2-1)}{x^2-1}+\frac{3}{x^2-1}=\frac{3x^2-3+3}{x^2-1}=\frac{3x^2}{x^2-1}.$$

9. $$\frac{(ax+4)(a+x)}{a+x}-\frac{ax^2+4x}{a+x}=\frac{a^2x+4a+ax^2+4x-(ax^2+4x)}{a+x}=\frac{a^2x+4a}{a+x}.$$

10. $$\frac{5bx}{a^2}-\frac{2a^2x}{a^2}=\frac{5bx-2a^2x}{a^2}.$$

11. $$\frac{a+b+c}{a+c}-\frac{a+c}{a+c}=\frac{a+b+c-(a+c)}{a+c}=\frac{b}{a+c}.$$

12. $$\frac{(3a-b)(a+1)}{a+1}-\frac{2a^2-b}{a+1}=\frac{3a^2-ab+3a-b-(2a^2-b)}{a+1}=\frac{a^2-ab+3a}{a+1}.$$

13. $$\frac{(x^5-1)(x-y)}{x-y}+\frac{x+y}{x-y}=\frac{(x^6-x-x^5y+y)+x+y}{x-y}=\frac{x^6-x^5y+2y}{x-y}.$$

14. $$\frac{(a+2b)(a-3b)}{a-3b}+\frac{ab+5b^2}{a-3b}=\frac{(a^2-ab-6b^2)+ab+5b^2}{a-3b}=\frac{a^2-b^2}{a-3b}.$$

15. $$\frac{a(m+n)(m-n)}{m-n}+\frac{an^2}{m-n}=\frac{(am^2-an^2)+an^2}{m-n}=\frac{am^2}{m-n}.$$

16. $$\frac{(a-x)(a+x)}{a+x}+\frac{a^2+x^2-5}{a+x}=\frac{(a^2-x^2)+a^2+x^2-5}{a+x}=\frac{2a^2-5}{a+x}.$$

17. $$\frac{(x^2-xy+y^2)(x^2+xy+y^2)}{x^2+xy+y^2}-\frac{x^4+y^4}{x^2+xy+y^2}=\frac{x^4+x^2y^2+y^4-(x^4+y^4)}{x^2+xy+y^2}$$
$$=\frac{x^2y^2}{x^2+xy+y^2}.$$

18. $$\frac{(x^2+2x-1)(x+5)}{x+5}-\frac{7x^2+9x}{x+5}=\frac{x^3+7x^2+9x-5-(7x^2+9x)}{x+5}=\frac{x^3-5}{x+5}.$$

19. $$\frac{(x^2+3x-5)(x^2+2x-1)}{x^2+2x-1}-\frac{5x^3-13x}{x^2+2x-1}=\frac{x^4+5x^3-13x+5-(5x^3-13x)}{x^2+2x-1}$$
$$=\frac{x^4+5}{x^2+2x-1}.$$

20. $$\frac{(x^2-x-1)(x^2+x+1)}{x^2+x+1}-\frac{x^4-2x^2-3x+2}{x^2+x+1}$$
$$=\frac{x^4-x^2-2x-1-(x^4-2x^2-3x+2)}{x^2+x+1}=\frac{x^2+x-3}{x^2+x+1}.$$

21. $$\frac{3a^4xy^2(2a^2x^2)}{2a^2x^2}=\frac{6a^6x^3y^2}{2a^2x^2}.$$

22. $$\frac{(a^2-x^2)(a^2+x^2)}{a^2+x^2}=\frac{a^4-x^4}{a^2+x^2}.$$

23. $$\frac{5x^2y^2z(4axz)}{4axz}=\frac{20ax^3y^2z^2}{4axz}.$$

24. $$\frac{ax(x+y)(x-y)}{x-y}=\frac{ax(x^2-y^2)}{x-y}=\frac{ax^3-axy^2}{x-y}.$$

25. $$\frac{(a^3+a^2b+ab^2+b^3)(a-b)}{a-b}=\frac{a^4-b^4}{a-b}.$$

Art. 164.

1. $$\frac{3x^2y^2}{9x^3y^4}=\frac{3x^2y^2}{3x^2y^2\times 3xy^2}=\frac{1}{3xy^2}.$$

2. $$\frac{14xy^2z^3}{49x^2yz^2}=\frac{7xyz^2\times 2yz}{7xyz^2\times 7x}=\frac{2yz}{7x}.$$

3. $$\frac{75x^5y^2z}{100x^3y^4z^2}=\frac{25x^3y^2z\times 3x^2}{25x^3y^2z\times 4y^2z}=\frac{3x^2}{4y^2z}.$$

4. $$\frac{35m^4n^2xz^2}{49pqm^3nzy}=\frac{7m^3nz\times 5mnxz}{7m^3nz\times 7pqy}=\frac{5mnxz}{7pqy}.$$

5. $$\frac{16a^4b^2}{72a^3bcde}=\frac{8a^3b\times 2ab}{8a^3b\times 9cde}=\frac{2ab}{9cde}.$$

6. $$\frac{6a^2x^3}{9a^3x^2}=\frac{3a^2x^2\times 2x}{3a^2x^2\times 3a}=\frac{2x}{3a}.$$

7. $$\frac{12x^4y^5z^3}{15x^5y^3z^2u^2}=\frac{3x^4y^3z^2\times 4y^2z}{3x^4y^3z^2\times 5xu^2}=\frac{4y^2z}{5xu^2}.$$

8. $$\frac{17a^2bc^3}{85a^3c^2x}=\frac{17a^2c^2\times bc}{17a^2c^2\times 5ax}=\frac{bc}{5ax}.$$

9. $$\frac{13abc^2xy^5}{65a^2b^2cx^4y}=\frac{13abcxy\times cy^4}{13abcxy\times 5abx^3}=\frac{cy^4}{5abx^3}.$$

10. $$\frac{a^2+ab}{2ab}=\frac{a(a+b)}{a\times 2b}=\frac{a+b}{2b}.$$

11. $$\frac{a^2+ab}{a^2-ab}=\frac{a(a+b)}{a(a-b)}=\frac{a+b}{a-b}.$$

12. $$\frac{5a^2x}{15a^2x-10ay^2}=\frac{5a\times ax}{5a(3ax-2y^2)}=\frac{ax}{3ax-2y^2}.$$

13. $$\frac{ax+bx}{a^2-b^2}=\frac{x(a+b)}{(a-b)(a+b)}=\frac{x}{a-b}.$$

14. $$\frac{3a^2+3ax}{a^2-x^2}=\frac{3a(a+x)}{(a-x)(a+x)}=\frac{3a}{a-x}.$$

15. $$\frac{(a+b)^2}{a^2-b^2}=\frac{(a+b)(a+b)}{(a+b)(a-b)}=\frac{a+b}{a-b}.$$

16. $$\frac{a^3+b^3}{a^2-b^2}=\frac{(a+b)(a^2-ab+b^2)}{(a+b)(a-b)}=\frac{a^2-ab+b^2}{a-b}.$$

17. $$\frac{x^3-xy^2}{x^4-y^4}=\frac{x(x^2-y^2)}{(x^2+y^2)(x^2-y^2)}=\frac{x}{x^2+y^2}.$$

18. $$\frac{x^2+3x+2}{x^2+6x+5}=\frac{(x+1)(x+2)}{(x+1)(x+5)}=\frac{x+2}{x+5}.$$

19. $$\frac{10a^2x}{5a^2x-15ay^2}=\frac{5a\times 2ax}{5a(ax-3y^2)}=\frac{2ax}{ax-3y^2}.$$

20. $$\frac{2(x+y)^2}{3(x^2-y^2)}=\frac{2(x+y)(x+y)}{3(x+y)(x-y)}=\frac{2(x+y)}{3(x-y)}.$$

21. $$\frac{2x^2+x-15}{2x^2-19x+35}=\frac{(2x-5)(x+3)}{(2x-5)(x-7)}=\frac{x+3}{x-7}.$$

22. $$\frac{x^2+10x+21}{x^2-2x-15}=\frac{(x+3)(x+7)}{(x+3)(x-5)}=\frac{x+7}{x-5}.$$

23. $$\frac{x^2+2x-3}{x^2+4x+3}=\frac{(x+3)(x-1)}{(x+3)(x+1)}=\frac{x-1}{x+1}.$$

24. $$\frac{x^4-x^2y^2}{x^4-y^4}=\frac{x^2(x^2-y^2)}{(x^2+y^2)(x^2-y^2)}=\frac{x^2}{x^2+y^2}.$$

25. $\dfrac{x^5+x^3}{x^4-1}=\dfrac{x^3(x^2+1)}{(x^2-1)(x^2+1)}=\dfrac{x^3}{x^2-1}$.

26. $\dfrac{x^2-7x+10}{x^2-2x-15}=\dfrac{(x-5)(x-2)}{(x-5)(x+3)}=\dfrac{x-2}{x+3}$.

27. $\dfrac{x^2+5x+6}{x^3+x+10}=\dfrac{(x+2)(x+3)}{(x+2)(x^2-2x+5)}=\dfrac{x+3}{x^2-2x+5}$.

28. $\dfrac{3x^2+23x-36}{4x^2+33x-27}=\dfrac{(x+9)(3x-4)}{(x+9)(4x-3)}=\dfrac{3x-4}{4x-3}$.

29. $\dfrac{a+3b}{17a^2-153b^2}=\dfrac{a+3b}{17(a^2-9b^2)}=\dfrac{a+3b}{17(a-3b)(a+3b)}=\dfrac{1}{17(a-3b)}$.

30. $\dfrac{x^3-ax^2}{x^2-2ax+a^2}=\dfrac{x^2(x-a)}{(x-a)(x-a)}=\dfrac{x^2}{x-a}$.

31. $\dfrac{axy}{bx^2y+cxy^2}=\dfrac{xy\times a}{xy(bx+cy)}=\dfrac{a}{bx+cy}$.

32. $\dfrac{3mn+3mr}{n^2-r^2}=\dfrac{3m(n+r)}{(n-r)(n+r)}=\dfrac{3m}{n-r}$.

33. $\dfrac{1+x+x^2}{1-x^3}=\dfrac{1+x+x^2}{(1+x+x^2)(1-x)}=\dfrac{1}{1-x}$.

34. $\dfrac{6x^2+x-15}{6x^2-11x-35}=\dfrac{(3x+5)(2x-3)}{(3x+5)(2x-7)}=\dfrac{2x-3}{2x-7}$.

35. $\dfrac{6m^2+mx-15x^2}{15m^2+16mx-15x^2}=\dfrac{(3m+5x)(2m-3x)}{(3m+5x)(5m-3x)}=\dfrac{2m-3x}{5m-3x}$.

36. $\dfrac{6x^2-11x-10}{6x^2-19x+10}=\dfrac{(3x+2)(2x-5)}{(3x-2)(2x-5)}=\dfrac{3x+2}{3x-2}$.

37. $\dfrac{2x^2+3x+1}{x^2-x-2}=\dfrac{(2x+1)(x+1)}{(x-2)(x+1)}=\dfrac{2x+1}{x-2}$.

38. $\dfrac{x^3+2x^2-5x-3\ \div\ \text{H. C. D. }(x^2+x-6)}{x^3-3x^2-10x+24\ \div\ \text{“}\qquad\text{“}}=\dfrac{x+1}{x-4}$.

39. $\dfrac{x^3-57x-56\ \div\ \text{H. C. D. }(x^2-7x-8)}{x^3+x^2-64x-64\ \div\ \text{“}\qquad\text{“}}=\dfrac{x+7}{x+8}$.

40. $\dfrac{9x^3-46x+35\ \div\ \text{H. C. D. }(3x^2+5x-7)}{15x^3+46x^2-49\ \div\ \text{“}\qquad\text{“}}=\dfrac{3x-5}{5x+7}$.

41. $\dfrac{4x^3-23x+5\ \div\ \text{H. C. D. } (2x^2-5x+1)}{10x^3-23x^2+1\ \div\ \quad\text{“}\qquad\text{“}} = \dfrac{2x+5}{5x+1}.$

42. $\dfrac{x^4-y^4}{x^8+x^6y^2+x^2y+y^3} = \dfrac{(x^2-y^2)(x^2+y^2)}{x^6(x^2+y^2)+y(x^2+y^2)} = \dfrac{(x^2-y^2)(x^2+y^2)}{(x^6+y)(x^2+y^2)}$

$$= \frac{x^2-y^2}{x^6+y}.$$

43. $\dfrac{x^3+x^2+x+1}{x^4-1} = \dfrac{x^2(x+1)+(x+1)}{(x^2+1)(x^2-1)} = \dfrac{(x^2+1)(x+1)}{(x^2+1)(x-1)(x+1)} = \dfrac{1}{x-1}.$

Art. 166.

7. L. C. M. of $a+b,\ a-b,\ a^2-b^2 = (a+b)(a-b) = a^2-b^2.$

$$\frac{a-b}{a+b} = \frac{(a-b)(a-b)}{(a+b)(a-b)} = \frac{(a-b)^2}{a^2-b^2},$$

$$\frac{a+2b}{a-b} = \frac{(a+2b)(a+b)}{(a-b)(a+b)} = \frac{(a+2b)(a+b)}{a^2-b^2},$$

$$\frac{a^2}{a^2-b^2} = \frac{a^2}{a^2-b^2}.$$

8. L. C. M. of $a^2+ab+b^2,\ a^3-b^3,\ a-b = (a^2+ab+b^2)(a-b) = a^3-b^3.$

$$\frac{a-b}{a^2+ab+b^2} = \frac{(a-b)(a-b)}{(a^2+ab+b^2)(a-b)} = \frac{(a-b)^2}{a^3-b^3},$$

$$\frac{a^2+b^2}{a^3-b^3} = \frac{a^2+b^2}{a^3-b^3},$$

$$\frac{b}{a-b} = \frac{b(a^2+ab+b^2)}{(a-b)(a^2+ab+b^2)} = \frac{b(a^2+ab+b^2)}{a^3-b^3}.$$

9. L. C. M. of $1+x,\ 4+4x,\ 1-x^2 = 4(1+x)(1-x) = 4(1-x^2).$

$$\frac{3}{1+x} = \frac{3\times4(1-x)}{4(1+x)(1-x)} = \frac{12(1-x)}{4(1-x^2)},$$

$$\frac{5}{4+4x} = \frac{5(1-x)}{(4+4x)(1-x)} = \frac{5(1-x)}{4(1-x^2)},$$

$$\frac{2x}{1-x^2} = \frac{4\times2x}{4(1-x^2)} = \frac{8x}{4(1-x^2)}.$$

10. L. C. M. of $x-y$, $x+y$, x^2-y^2, $x^2+y^2 = (x-y)(x+y)(x^2+y^2)$
$= x^4-y^4.$

$$\frac{x}{x-y} = \frac{x(x+y)(x^2+y^2)}{(x-y)(x+y)(x^2+y^2)} = \frac{x(x^3+x^2y+xy^2+y^3)}{x^4-y^4},$$

$$\frac{y}{x+y} = \frac{y(x-y)(x^2+y^2)}{(x+y)(x-y)(x^2+y^2)} = \frac{y(x^3-x^2y+xy^2-y^3)}{x^4-y^4},$$

$$\frac{xy}{x^2-y^2} = \frac{xy(x^2+y^2)}{(x^2-y^2)(x^2+y^2)} = \frac{xy(x^2+y^2)}{x^4-y^4},$$

$$\frac{y^2}{x^2+y^2} = \frac{y^2(x-y)(x+y)}{(x^2+y^2)(x-y)(x+y)} = \frac{y^2(x^2-y^2)}{x^4-y^4}.$$

11. L. C. M. of a^2-ab+b^2, a^2+ab+b^2, $a^4+a^2b^2+b^4$
$= (a^2-ab+b^2)(a^2+ab+b^2) = a^4+a^2b^2+b^4.$

$$\frac{a}{a^2-ab+b^2} = \frac{a(a^2+ab+b^2)}{(a^2-ab+b^2)(a^2+ab+b^2)} = \frac{a(a^2+ab+b^2)}{a^4+a^2b^2+b^4},$$

$$\frac{b}{a^2+ab+b^2} = \frac{b(a^2-ab+b^2)}{(a^2+ab+b^2)(a^2-ab+b^2)} = \frac{b(a^2-ab+b^2)}{a^4+a^2b^2+b^4},$$

$$\frac{c}{a^4+a^2b^2+b^4} = \frac{c}{a^4+a^2b^2+b^4}.$$

12. L. C. M. of $x^2-8x+15$, x^2-4x-5, x^2+4x+3
$= (x-5)(x-3)(x+1)(x+3).$

$$\frac{x-1}{x^2-8x+15} = \frac{(x-1)(x+1)(x+3)}{(x^2-8x+15)(x+1)(x+3)} = \frac{(x^2-1)(x+3)}{(x-5)(x-3)(x+1)(x+3)},$$

$$\frac{x+3}{x^2-4x-5} = \frac{(x+3)(x-3)(x+3)}{(x^2-4x-5)(x-3)(x+3)} = \frac{(x+3)(x^2-9)}{(x-5)(x+1)(x-3)(x+3)},$$

$$\frac{x-5}{x^2+4x+3} = \frac{(x-5)(x-5)(x-3)}{(x^2+4x+3)(x-5)(x-3)} = \frac{(x-5)^2(x-3)}{(x+3)(x+1)(x-5)(x-3)}.$$

13. L. C. M. of x^2-4, x^2+x-6, $x^2+6x+9 = (x-2)(x+2)(x+3)(x+3)$
$= (x^2-4)(x+3)^2.$

$$\frac{x-3}{x^2-4}=\frac{(x-3)(x+3)(x+3)}{(x^2-4)(x+3)(x+3)}=\frac{(x^2-9)(x+3)}{(x^2-4)(x+3)^2},$$

$$\frac{x-2}{x^2+x-6}=\frac{(x-2)(x+2)(x+3)}{(x^2+x-6)(x+2)(x+3)}=\frac{(x^2-4)(x+3)}{(x^2-4)(x+3)^2},$$

$$\frac{x^2+4}{x^2+6x+9}=\frac{(x^2+4)(x-2)(x+2)}{(x+3)^2(x-2)(x+2)}=\frac{x^4-16}{(x^2-4)(x+3)^2}.$$

14. L. C. M. of x^2+x-20, x^2-x-12, $x^2+x-6=(x+5)(x-4)(x+3)(x-2)$
$=(x^2+x-20)(x^2+x-6)$.

$$\frac{5}{x^2+x-20}=\frac{5(x^2+x-6)}{(x^2+x-20)(x^2+x-6)},$$

$$\frac{3}{x^2-x-12}=\frac{3(x+5)(x-2)}{(x^2-x-12)(x+5)(x-2)}=\frac{3(x^2+3x-10)}{(x^2+x-20)(x^2+x-6)},$$

$$\frac{2}{x^2+x-6}=\frac{2(x^2+x-20)}{(x^2+x-20)(x^2+x-6)}.$$

15. L. C. M. of $(a-x)(a-y)$, $(a-y)(b-x)=(a-x)(a-y)(b-x)$.

$$\frac{b+x}{(a-x)(a-y)}=\frac{(b+x)(b-x)}{(a-x)(a-y)(b-x)}=\frac{b^2-x^2}{(a-x)(a-y)(b-x)},$$

$$\frac{a+x}{(a-y)(b-x)}=\frac{(a+x)(a-x)}{(a-y)(b-x)(a-x)}=\frac{a^2-x^2}{(a-x)(a-y)(b-x)}.$$

Art. 172.

10. $$\frac{am+b}{m^2-n^2}-\frac{a}{m+n}=\frac{am+b}{(m+n)(m-n)}-\frac{a(m-n)}{(m+n)(m-n)}$$
$$=\frac{am+b-(am-an)}{m^2-n^2}=\frac{an+b}{m^2-n^2}.$$

11. $$\frac{x+y}{x-y}-\frac{x-y}{x+y}=\frac{(x+y)(x+y)}{x^2-y^2}-\frac{(x-y)(x-y)}{x^2-y^2}=\frac{4xy}{x^2-y^2}.$$

12. $$\frac{a+n}{x+y}+\frac{a-n}{x-y}=\frac{(a+n)(x-y)}{x^2-y^2}+\frac{(a-n)(x+y)}{x^2-y^2}=\frac{2(ax-ny)}{x^2-y^2}.$$

14. $\dfrac{1}{m-1}-\dfrac{m}{m^2-1}=\dfrac{m+1}{m^2-1}-\dfrac{m}{m^2-1}=\dfrac{1}{m^2-1}.$

15. $\dfrac{3-8x}{x^2-9}-\dfrac{x^2+3x+1}{x+3}=\dfrac{3-8x}{x^2-9}-\dfrac{(x^2+3x+1)(x-3)}{x^2-9}=\dfrac{6-x^3}{x^2-9}.$

17. $\dfrac{1+3x}{1-3x}-\dfrac{1-3x}{1+3x}=\dfrac{(1+3x)^2}{1-9x^2}-\dfrac{(1-3x)^2}{1-9x^2}=\dfrac{12x}{1-9x^2}.$

21. $\dfrac{x}{x-y}-\dfrac{y}{y-x}=\dfrac{x}{x-y}+\dfrac{y}{x-y}=\dfrac{x+y}{x-y}.$

22. $\dfrac{1}{x-a}+\dfrac{1}{x+a}-\dfrac{2}{x}=\dfrac{x(x+a)+x(x-a)-2(x^2-a^2)}{x(x^2-a^2)}=\dfrac{2a^2}{x(x^2-a^2)}.$

23. $\dfrac{a}{a-x}+\dfrac{3a}{a+x}-\dfrac{2ax}{a^2-x^2}=\dfrac{a(a+x)+3a(a-x)-2ax}{a^2-x^2}=\dfrac{4a(a-x)}{a^2-x^2}$

$=\dfrac{4a}{a+x}.$

24. $\dfrac{n(4y+1)}{x^2-y^2}+\dfrac{m+2n}{x+y}+\dfrac{m-2n}{x-y}$

$=\dfrac{n(4y+1)+(m+2n)(x-y)+(m-2n(x+y)}{x^2-y^2}=\dfrac{2mx+n}{x^2-y^2}.$

25. $\dfrac{1+x}{x+3}+\dfrac{1-x}{x-2}+\dfrac{2x-1}{x+2}$

$=\dfrac{(x+1)(x^2-4)+(1-x)(x^2+5x+6)+(2x-1)(x^2+x-6)}{(x+3)(x-2)(x+2)}$

$=\dfrac{2x^3-2x^2-18x+8}{x^3+3x^2-4x-12}.$

26. $\dfrac{a-2b}{3c}-\dfrac{b-3c}{2a}+\dfrac{4ab+3bc}{6ac}=\dfrac{2a(a-2b)-3c(b-3c)+(4ab+3bc)}{6ac}$

$=\dfrac{2a^2+9c^2}{6ac}.$

27. $\dfrac{3a+7}{13}-\dfrac{a-3}{3}+\dfrac{2(2a+9)}{39}=\dfrac{3(3a+7)-13(a-3)+2(2a+9)}{39}$

$=\dfrac{78}{39}=2.$

28. $\dfrac{2m}{m^2-n^2}+\dfrac{1}{m+n}-\dfrac{1}{m-n}=\dfrac{2m+(m-n)-(m+n)}{m^2-n^2}=\dfrac{2(m-n)}{m^2-n^2}$
$=\dfrac{2}{m+n}.$

29. $\dfrac{1}{a+b}+\dfrac{b}{a^2-b^2}-\dfrac{a}{a^2+b^2}=\dfrac{(a-b)(a^2+b^2)+b(a^2+b^2)-a(a^2-b^2)}{(a+b)(a-b)(a^2+b^2)}$
$=\dfrac{2ab^2}{a^4-b^4}.$

30. $\dfrac{2}{x}-\dfrac{3}{2x-1}-\dfrac{2x-3}{4x^2-1}=\dfrac{2(4x^2-1)-3x(2x+1)-x(2x-3)}{x(4x^2-1)}=\dfrac{-2}{x(4x^2-1)}$
$=\dfrac{2}{x(1-4x^2)}.$

31. $\dfrac{2b-a}{x-b}+\dfrac{b-2a}{x+b}+\dfrac{3x(a-b)}{x^2-b^2}=\dfrac{(2b-a)(x+b)+(b-2a)(x-b)+3x(a-b)}{x^2-b^2}$
$=\dfrac{ab+b^2}{x^2-b^2}=\dfrac{b(a+b)}{x^2-b^2}.$

32. $\dfrac{3}{x}-\dfrac{5}{2x-1}-\dfrac{2x-7}{4x^2-1}=\dfrac{3(4x^2-1)-5x(2x+1)-x(2x-7)}{x(4x^2-1)}$
$=\dfrac{2x-3}{x(4x^2-1)}.$

33. $\dfrac{1}{x+1}-\dfrac{2}{x+2}+\dfrac{1}{x+3}=\dfrac{(x+2)(x+3)-2(x+1)(x+3)+(x+1)(x+2)}{(x+1)(x+2)(x+3)}$
$=\dfrac{2}{(x+1)(x+2)(x+3)}.$

34. $\dfrac{3}{2x-4}-\dfrac{1}{x+2}-\dfrac{x+10}{2x^2+8}$
$=\dfrac{3(x+2)(x^2+4)-2(x-2)(x^2+4)-(x+10)(x^2-4)}{2(x^2-4)(x^2+4)}$
$=\dfrac{8x+80}{2(x^2-4)(x^2+4)}=\dfrac{4x+40}{(x^2-4)(x^2+4)}=\dfrac{4(x+10)}{x^4-16}.$

35. $\dfrac{2}{x+4}-\dfrac{x-3}{x^2-4x+16}+\dfrac{x^2}{x^3+64}=\dfrac{2(x^2-4x+16)-(x-3)(x+4)+x^2}{x^3+64}$
$=\dfrac{2x^2-9x+44}{x^3+64}.$

36. $\dfrac{a(2a-b)}{a^3+b^3}-\dfrac{a-b}{a^2-ab+b^2}=\dfrac{a(2a-b)-(a-b)(a+b)}{a^3+b^3}=\dfrac{a^2-ab+b^2}{a^3+b^3}$

$$=\frac{1}{a+b}.$$

37. $\dfrac{a^4+3a^2b^2+b^4}{a^3-b^3}-\dfrac{a^2-ab+b^2}{a-b}=\dfrac{a^4+3a^2b^2+b^4-(a^2-ab+b^2)(a^2+ab+b^2)}{a^3-b^3}$

$$=\frac{2a^2b^2}{a^3-b^3}.$$

38. $\dfrac{2x+14}{x^3+x^2-49x-49}-\dfrac{1-x}{x^2-6x-7}=\dfrac{2(x+7)}{x^2(x+1)-49(x+1)}-\dfrac{1-x}{(x-7)(x+1)}$

$$=\frac{2(x+7)-(1-x)(x+7)}{(x+7)(x-7)(x+1)}=\frac{(1+x)(x+7)}{(x+7)(x-7)(x+1)}=\frac{1}{x-7}.$$

39. $\dfrac{2x^2}{x^4-y^4}-\dfrac{x+y}{x^3+x^2y+xy^2+y^3}=\dfrac{2x^2-(x+y)(x-y)}{x^4-y^4}$

$$=\frac{x^2+y^2}{x^4-y^4}=\frac{1}{x^2-y^2}.$$

40. $\dfrac{1}{x-1}-\dfrac{1}{x+2}-\dfrac{3}{(x+2)^2}=\dfrac{(x+2)^2-(x+2)(x-1)-3(x-1)}{(x-1)(x+2)^2}$

$$=\frac{9}{(x-1)(x+2)^2}.$$

41. $\dfrac{(a^2+b^2)^2}{ab(a-b)^2}-\dfrac{a}{b}-\dfrac{b}{a}-2=\dfrac{(a^2+b^2)^2-a^2(a-b)^2-b^2(a-b)^2-2ab(a-b)^2}{ab(a-b)^2}$

$$=\frac{(a^2+b^2)^2-(a^2+b^2+2ab)(a-b)^2}{ab(a-b)^2}=\frac{4a^2b^2}{ab(a-b)^2}=\frac{4ab}{(a-b)^2}.$$

42. $\dfrac{x}{x-y}+\dfrac{3x}{x+y}+\dfrac{2xy}{y^2-x^2}=\dfrac{x}{x-y}+\dfrac{3x}{x+y}-\dfrac{2xy}{x^2-y^2}$

$$=\frac{x(x+y)+3x(x-y)-2xy}{x^2-y^2}=\frac{4x^2-4xy}{x^2-y^2}=\frac{4x(x-y)}{x^2-y^2}=\frac{4x}{x+y}.$$

NOTE.—Changing y^2-x^2 to x^2-y^2 changes the sign of the fraction.

43. $$\frac{a}{b-a}-\frac{3a}{a+b}-\frac{2ab}{a^2-b^2}=\frac{-a}{a-b}-\frac{3a}{a+b}-\frac{2ab}{a^2-b^2}$$

$$=\frac{-a(a+b)-3a(a-b)-2ab}{a^2-b^2}=\frac{-4a^2}{a^2-b^2}=\frac{4a^2}{b^2-a^2}.$$

44. $$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}$$

$$=\frac{a^2}{(a-b)(a-c)}-\frac{b^2}{(a-b)(b-c)}-\frac{c^2}{(a-c)(b-c)}$$

$$=\frac{a^2(b-c)-b^2(a-c)-c^2(a-b)}{(a-b)(a-c)(b-c)}=1.$$

NOTE.—Changing $b-a$ to $a-b$, and $c-a$ to $a-c$, changes the sign of each fraction.

45. $$\frac{3}{1-2x}-\frac{7}{1+2x}-\frac{4-20x}{4x^2-1}=\frac{3}{1-2x}-\frac{7}{1+2x}+\frac{4-20x}{1-4x^2}$$

$$=\frac{3(1+2x)-7(1-2x)+4-20x}{1-4x^2}=\frac{0}{1-4x^2}=0.$$

NOTE.—Changing $4x^2-1$ to $1-4x^2$ changes the sign of the fraction.

46. $$\frac{c}{(x-a)(a-b)}-\frac{c}{(b-x)(b-a)}=\frac{c}{(x-a)(a-b)}-\frac{c}{(x-b)(a-b)}$$

$$=\frac{c(x-b)-c(x-a)}{(x-a)(x-b)(a-b)}=\frac{c(a-b)}{(x-a)(x-b)(a-b)}=\frac{c}{(x-a)(x-b)}.$$

NOTE.—Changing both $b-x$ and $b-a$ does not affect the sign of the fraction.

47. $$\left\{\frac{1}{x^3}+\frac{1}{x^2}+\frac{x-1}{x^2+1}\right\}-\left\{\frac{1}{x}+\frac{1}{(x^2+1)^2}\right\}$$

$$=\left\{\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}\right\}+\left\{\frac{x-1}{x^2+1}-\frac{1}{(x^2+1)^2}\right\}$$

$$=\frac{1+x-x^2}{x^3}+\frac{(x-1)(x^2+1)-1}{(x^2+1)^2}$$

$$=\frac{(1+x-x^2)(x^2+1)^2+x^3(x-1)(x^2+1)-x^3}{x^3(x^2+1)^2}=\frac{x^2+x+1}{x^3(x^2+1)^2}.$$

48. $$\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$$

$$=\frac{25(x-1)(2x+3)-(x+1)(2x+3)-48(x+1)(x-1)}{10(x+1)(x-1)(2x+3)}$$

$$=\frac{20x-30}{10(x+1)(x-1)(2x+3)}=\frac{2x-3}{(x^2-1)(2x+3)}.$$

49. $$\left\{\frac{2}{a-b}+\frac{2}{b-c}-\frac{2}{a-c}\right\}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(c-b)(a-c)}$$

$$=\frac{2(b-c)(a-c)+2(a-b)(a-c)-2(a-b)(b-c)}{(a-b)(b-c)(a-c)}$$

$$-\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(a-c)}=\frac{0}{(a-b)(b-c)(a-c)}=0.$$

50. $$\frac{a^3+b^3}{ab(a+b)}-\left\{\frac{a}{b}+\frac{b}{a}+2\right\}=\frac{a^3+b^3}{ab(a+b)}-\frac{a^2+b^2+2ab}{ab}$$

$$=\frac{a^3+b^3-(a^2+2ab+b^2)(a+b)}{ab(a+b)}=\frac{-3ab(a+b)}{ab(a+b)}=-3.$$

51. $$\frac{1}{(x-3)(x-4)}+\frac{2}{(4-x)(x-2)}+\frac{1}{(2-x)(3-x)}$$

$$=\frac{1}{(x-3)(x-4)}-\frac{2}{(x-4)(x-2)}+\frac{1}{(x-2)(x-3)}$$

$$=\frac{(x-2)-2(x-3)+x-4}{(x-3)(x-4)(x-2)}=\frac{0}{(x-3)(x-4)(x-2)}=0.$$

52. $$\frac{1}{a-2b}-\frac{4}{a-b}+\frac{6}{a}-\frac{4}{a+b}+\frac{1}{a+2b}$$

$$=\left\{\frac{1}{a-2b}+\frac{1}{a+2b}\right\}-\left\{\frac{4}{a-b}+\frac{4}{a+b}\right\}+\frac{6}{a}$$

$$=\frac{2a^2(a^2-b^2)-8a^2(a^2-4b^2)+6(a^2-4b^2)(a^2-b^2)}{a(a^2-b^2)(a^2-4b^2)}$$

$$=\frac{24b^4}{a(a^2-b^2)(a^2-4b^2)}.$$

Art. 175.

11. $\dfrac{ax}{a+x} \times \left(\dfrac{x}{a} - \dfrac{a}{x}\right) = \dfrac{ax}{a+x} \times \dfrac{x^2-a^2}{ax} = \dfrac{ax(x+a)(x-a)}{ax(x+a)} = x-a.$

12. $\left(b + \dfrac{a^2}{b}\right) \times \left(a - \dfrac{b^2}{a}\right) = \dfrac{a^2+b^2}{b} \times \dfrac{a^2-b^2}{a} = \dfrac{a^4-b^4}{ab}.$

13. $\dfrac{(a-b)^2}{b+a} \times \dfrac{b}{x(a-b)} = \dfrac{b(a-b)(a-b)}{x(a-b)(a+b)} = \dfrac{b(a-b)}{x(a+b)}.$

14. $\left(a - \dfrac{x^2}{a}\right) \times \left(\dfrac{a}{x} + \dfrac{x}{a}\right) = \dfrac{a^2-x^2}{a} \times \dfrac{a^2+x^2}{ax} = \dfrac{a^4-x^4}{a^2x}.$

15. $\dfrac{a^2-b^2}{a^3-a^2b} \times \dfrac{a^2b}{a^3+b^3} = \dfrac{a^2b(a-b)(a+b)}{a^2(a-b)(a+b)(a^2-ab+b^2)} = \dfrac{b}{a^2-ab+b^2}.$

16. $\dfrac{x^2+xy}{x^2+y^2} \times \dfrac{x^3-y^3}{xy(x+y)} = \dfrac{x(x+y)(x^3-y^3)}{xy(x+y)(x^2+y^2)} = \dfrac{x^3-y^3}{y(x^2+y^2)}.$

17. $\dfrac{x+1}{x-1} \times \dfrac{x+2}{x^2-1} \times \dfrac{x-1}{(x+2)^2} = \dfrac{(x+1)(x+2)(x-1)}{(x-1)(x+1)(x-1)(x+2)^2} = \dfrac{1}{(x-1)(x+2)}.$

18. $\dfrac{27+x^3}{y^3-8} \times \dfrac{y^2+2y+4}{x^2-3x+9} = \dfrac{(3+x)(9-3x+x^2)(y^2+2y+4)}{(y-2)(y^2+2y+4)(9-3x+x^2)} = \dfrac{3+x}{y-2}.$

19. $\dfrac{1-x^2}{1+y} \times \dfrac{1-y^2}{x+x^2} \times \left(1 + \dfrac{x}{1-x}\right) = \dfrac{(1-x)(1+x)(1-y)(1+y)}{(1+y)(1-x)(1+x)x} = \dfrac{1-y}{x}.$

20. $\dfrac{3ax}{4by} \times \dfrac{a^2-x^2}{c^2-x^2} \times \dfrac{bc+bx}{a^2+ax} \times \dfrac{c-x}{a-x} = \dfrac{3ax(a-x)(a+x)(c-x)(c+x)b}{4by(c-x)(c+x)(a-x)(a+x)a}$

$$= \dfrac{3abx}{4aby} = \dfrac{3x}{4y}.$$

21. $\left(b + \dfrac{bx}{a}\right) \times \left(1 - \dfrac{a}{a+x}\right) \times \dfrac{a}{bx} = \dfrac{ab+bx}{a} \times \dfrac{x}{a+x} \times \dfrac{a}{bx}$

$$= \dfrac{abx(a+x)}{abx(a+x)} = 1.$$

22. $\dfrac{a^3-64}{x^3+216} \times \dfrac{x^2-6x+36}{a^2+4a+16} = \dfrac{(a-4)(a^2+4a+16)(x^2-6x+36)}{(x+6)(x^2-6x+36)(a^2+4a+16)} = \dfrac{a-4}{x+6}.$

23. $\dfrac{a^2-b^2}{a^3+b^3} \times \dfrac{a^4-b^4}{(a+b)^3} \times \dfrac{a+b}{(a-b)^3} = \dfrac{(a-b)(a+b)(a+b)(a-b)(a^2+b^2)(a+b)}{(a^2+b^2)(a+b)^2(a-b)^2}$

$= a+b.$

24. $\dfrac{x(a-x)}{a^2+2ax+x^2} \times \dfrac{a(a+x)}{a^2-2ax+x^2} = \dfrac{ax(a+x)(a-x)}{(a+x)(a+x)(a-x)(a-x)} = \dfrac{ax}{a^2-x^2}.$

25. $\dfrac{a^4-b^4}{a^2-2ab+b^2} \times \dfrac{a-b}{a^2+ab} = \dfrac{(a^2+b^2)(a-b)(a+b)(a-b)}{(a-b)(a-b)(a+b)a} = \dfrac{a^2+b^2}{a}.$

26. $\dfrac{a^2-x^2}{m^2+mn+n^2} \times \dfrac{m^3-n^3}{a^3+x^3} \times \dfrac{a^2-ax+x^2}{m^2-n^2}$

$= \dfrac{(a-x)(a+x)(m-n)(m^2+mn+n^2)(a^2-ax+x^2)}{(m^2+mn+n^2)(a+x)(a^2-ax+x^2)(m-n)(m+n)} = \dfrac{a-x}{m+n}.$

27. $\dfrac{x^2+2x+4}{x^2-2x+4} \times \dfrac{x^3+8}{x^3-8} = \dfrac{(x^2+2x+4)(x+2)(x^2-2x+4)}{(x^2-2x+4)(x-2)(x^2+2x+4)} = \dfrac{x+2}{x-2}.$

28. $\left(\dfrac{x^2}{a^2} - \dfrac{x}{a} + 1\right) \times \left(\dfrac{x^2}{a^2} + \dfrac{x}{a} + 1\right) = \dfrac{x^2-ax+a^2}{a^2} \times \dfrac{x^2+ax+a^2}{a^2}$

$= \dfrac{x^4+a^2x^2+a^4}{a^4} = \dfrac{x^4}{a^4} + \dfrac{x^2}{a^2} + 1.$

29. $\dfrac{x^2-9x+20}{x^2-6x} \times \dfrac{x^2-13x+42}{x^2-5x} = \dfrac{(x-5)(x-4)(x-7)(x-6)}{x^2(x-6)(x-5)}$

$= \dfrac{(x-4)(x-7)}{x^2} = \dfrac{x^2-11x+28}{x^2}.$

30. $\left(1 - \dfrac{a-b}{a+b}\right) \times \left(2 + \dfrac{2b}{a-b}\right) = \dfrac{2b}{a+b} \times \dfrac{2a}{a-b} = \dfrac{4ab}{a^2-b^2}.$

31. $\dfrac{a^2-x^2}{a+b} \times \dfrac{a^2-b^2}{ax+x^2} \times \left(a + \dfrac{ax}{a-x}\right) = \dfrac{(a-x)(a+x)(a-b)(a+b)a^2}{(a+b)(a-x)(a+x)x}$

$= \dfrac{a^2(a-b)}{x}.$

32. $\dfrac{x(a-x)}{a^2+2ax+x^2} \times \dfrac{x(a+x)}{a^2-2ax+x^2} = \dfrac{x^2(a-x)(a+x)}{(a+x)(a+x)(a-x)(a-x)} = \dfrac{x^2}{a^2-x^2}.$

33. $\dfrac{1-x^2}{1+y} \times \dfrac{1-y^2}{x+x^2} \times \left\{1 - \dfrac{x}{x-1}\right\} = \dfrac{(1-x)(1+x)(1-y)(1+y)}{(1+y)(1+x)x} \times \dfrac{1}{1-x}$

$$= \frac{(1-x)(1+x)(1-y)(1+y)}{x(1+y)(1+x)(1-x)} = \frac{1-y}{x}.$$

34. $\left\{a - \dfrac{a^2+x^2}{a}\right\} \times \left\{x + \dfrac{a^2-x^2}{x}\right\} = \dfrac{-x^2}{a} \times \dfrac{a^2}{x} = \dfrac{-a^2x^2}{ax} = -ax.$

35. $\dfrac{x^2+2x-3}{x^2+5x+6} \times \dfrac{4x^2-12x-40}{3x^2-18x+15} = \dfrac{(x+3)(x-1)(x-5)(x+2)4}{(x+3)(x+2)(x-5)(x-1)3} = \dfrac{4}{3}.$

36. $\left\{\dfrac{x^2}{x^2-y^2} - \dfrac{y^2}{x^2+y^2}\right\} \times \dfrac{(x^2-y^2)^2}{(x^2-y^2)^2 + (x^2+y^2)^2} = \dfrac{x^4+y^4}{x^4-y^4} \times \dfrac{(x^2-y^2)^2}{2x^4+2y^4}$

$$= \frac{(x^4+y^4)(x^2-y^2)(x^2-y^2)}{(x^2+y^2)(x^2-y^2)(x^4+y^4)2} = \frac{x^2-y^2}{2(x^2+y^2)}.$$

37. $\dfrac{a^2-ab+b^2}{a^3-3ab(a-b)-b^3} \times \dfrac{a^2-b^2}{a^3+b^3} = \dfrac{(a^2-ab+b^2)(a-b)(a+b)}{(a-b)(a-b)(a-b)(a+b)(a^2-ab+b^2)}$

$$= \frac{1}{(a-b)^2}.$$

Art. 177.

5. $\dfrac{a^2b}{a^2-b^2} \div \dfrac{ab^2}{a^2+ab} = \dfrac{a^2b}{a^2-b^2} \times \dfrac{a^2+ab}{ab^2} = \dfrac{a^3b(a+b)}{ab^2(a+b)(a-b)} = \dfrac{a^2}{b(a-b)}.$

6. $\dfrac{x^3-y^3}{(x^2+y^2)^2} \div \dfrac{x-y}{x^2+y^2} = \dfrac{x^3-y^3}{(x^2+y^2)^2} \times \dfrac{x^2+y^2}{x-y} = \dfrac{(x-y)(x^2+xy+y^2)(x^2+y^2)}{(x^2+y^2)(x^2+y^2)(x-y)}$

$$= \frac{x^2+xy+y^2}{x^2+y^2}.$$

7. $\dfrac{2y^2}{x^3+y^3} \div \dfrac{y}{y+x} = \dfrac{2y^2}{x^3+y^3} \times \dfrac{x+y}{y} = \dfrac{2y^2(x+y)}{y(x+y)(x^2-xy+y^2)}$

$$= \frac{2y}{x^2-xy+y^2}.$$

8. $\dfrac{ax-x^2}{(a+x)^2} \div \dfrac{x^2}{a^2-x^2} = \dfrac{ax-x^2}{(a+x)^2} \times \dfrac{a^2-x^2}{x^2} = \dfrac{x(a-x)(a-x)(a+x)}{(a+x)(a+x)x^2}$

$$= \frac{(a-x)^2}{x(a+x)}.$$

9. $$\frac{(x^2-y^2)\,xy}{(a+b)^2} \div \frac{xy^2\,(x+y)}{a^2+ab} = \frac{(x^2-y^2)\,xy}{(a+b)^2} \times \frac{a^2+ab}{xy^2\,(x+y)}$$

$$= \frac{axy\,(x+y)\,(x-y)\,(a+b)}{(a+b)\,(a+b)\,(x+y)\,xy^2} = \frac{a\,(x-y)}{y\,(a+b)}.$$

10. $$\frac{x^2-5x+6}{x^2-5x} \div \frac{x^2-3x}{x^2-6x+5} = \frac{x^2-5x+6}{x^2-5x} \times \frac{x^2-6x+5}{x^2-3x}$$

$$= \frac{(x-3)\,(x-2)\,(x-5)\,(x-1)}{x^2\,(x-5)\,(x-3)} = \frac{(x-2)\,(x-1)}{x^2} = \frac{x^2-3x+2}{x^2}.$$

11. $$\frac{x^4-y^4}{x^2-2xy+y^2} \div \frac{x^2+xy}{x-y} = \frac{x^4-y^4}{x^2-2xy+y^2} \times \frac{x-y}{x^2+xy}$$

$$= \frac{(x^2+y^2)\,(x-y)\,(x+y)\,(x-y)}{(x-y)\,(x-y)\,(x+y)\,x} = \frac{x^2+y^2}{x}.$$

12. $$\frac{2m^2n}{m^2-n^2} \div \frac{3mn^2}{m^2+mn} = \frac{2m^2n}{m^2-n^2} \times \frac{m^2+mn}{3mn^2} = \frac{2m^3n\,(m+n)}{3mn^2\,(m+n)\,(m-n)}$$

$$= \frac{2m^2}{3n\,(m-n)}.$$

13. $$\frac{x^4-y^4}{a^2b+ab^2} \times \frac{a+b}{(x+y)^2} \div \frac{(x-y)^2}{ab} = \frac{x^4-y^4}{a^2b+ab^2} \times \frac{a+b}{(x+y)^2} \times \frac{ab}{(x-y)^2}$$

$$= \frac{(x^2+y^2)\,(x-y)\,(x+y)\,(a+b)\,ab}{ab\,(a+b)\,(x+y)\,(x+y)\,(x-y)\,(x-y)} = \frac{x^2+y^2}{x^2-y^2}.$$

14. $$\left\{\frac{x^2}{y^3} + \frac{1}{x}\right\} \div \left\{\frac{x}{y^2} - \frac{1}{y} + \frac{1}{x}\right\} = \frac{x^3+y^3}{xy^3} \div \frac{x^2-xy+y^2}{xy^2}$$

$$= \frac{x^3+y^3}{xy^3} \times \frac{xy^2}{x^2-xy+y^2} = \frac{xy^2\,(x+y)\,(x^2-xy+y^2)}{xy^3\,(x^2-xy+y^2)} = \frac{x+y}{y}.$$

15. $$\left(\frac{x+y}{x-y} + \frac{x-y}{x+y}\right) \div \left(\frac{x+y}{2x-2y} - \frac{x-y}{2x+2y}\right)$$

$$= \frac{(x+y)^2+(x-y)^2}{x^2-y^2} \div \frac{2\,(x+y)^2-2\,(x-y)^2}{4\,(x^2-y^2)} = \frac{2\,(x^2+y^2)}{x^2-y^2} \times \frac{4(x^2-y^2)}{8xy}$$

$$= \frac{8\,(x^2+y^2)\,(x^2-y^2)}{8xy\,(x^2-y^2)} = \frac{x^2+y^2}{xy}.$$

16. $\dfrac{x^2-6x+8}{x^2-4x+3}\times\dfrac{x^2-5x+6}{x^2-2x+8}\div\dfrac{(x-2)^2}{x-1}$

$$=\frac{(x-4)(x-2)(x-3)(x-2)(x-1)}{(x-3)(x-1)(x^2-2x+8)(x-2)(x-2)}=\frac{x-4}{x^2-2x+8}.$$

Art. 179.

3. $\dfrac{\dfrac{(x^2-y^2)xy}{(a+b)^2}}{\dfrac{(x+y)xy^2}{a^2+ab}}=\dfrac{(x^2-y^2)xy}{(a+b)^2}\times\dfrac{a^2+ab}{(x+y)xy^2}=\dfrac{axy(x-y)(x+y)(a+b)}{xy^2(a+b)(a+b)(x+y)}$

$$=\frac{a(x-y)}{y(a+b)}.$$

4. $\dfrac{\dfrac{a^2b}{a^2-b^2}}{\dfrac{ab^2}{a^2+ab}}=\dfrac{a^2b}{a^2-b^2}\times\dfrac{a^2+ab}{ab^2}=\dfrac{a^2b(a+b)}{ab^2(a+b)(a-b)}=\dfrac{a^2}{b(a-b)}.$

5. $\dfrac{\dfrac{x^4-y^4}{x^2-2xy+y^2}}{\dfrac{x^2+xy}{x-y}}=\dfrac{x^4-y^4}{x^2-2xy+y^2}\times\dfrac{x-y}{x^2+xy}=\dfrac{(x^2+y^2)(x-y)(x+y)(x-y)}{(x-y)(x-y)(x+y)x}$

$$=\frac{x^2+y^2}{x}.$$

6. $\dfrac{\dfrac{x^2}{2a^2}+\dfrac{6a^2}{x^2}-4}{\dfrac{x}{2a}-\dfrac{3a}{x}}=\dfrac{x^4+12a^4-8a^2x^2}{2a^2x^2}\div\dfrac{x^2-6a^2}{2ax}$

$$=\frac{x^4-8a^2x^2+12a^4}{2a^2x^2}\times\frac{2ax}{x^2-6a^2}=\frac{(x^2-6a^2)(x^2-2a^2)2ax}{(x^2-6a^2)2a^2x^2}$$

$$=\frac{x^2-2a^2}{ax}.$$

7. $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}=\dfrac{2ac-2bd}{c^2-d^2}\div\dfrac{2ac+2bd}{c^2-d^2}=\dfrac{2(ac-bd)}{c^2-d^2}\times\dfrac{c^2-d^2}{2(ac+bd)}$

$$=\frac{2(ac-bd)(c^2-d^2)}{2(ac+bd)(c^2-d^2)}=\frac{ac-bd}{ac+bd}.$$

8. $$\frac{\dfrac{x^2-5x+6}{x^2-5x}}{\dfrac{x^2-3x}{x^2-6x+5}} = \frac{x^2-5x+6}{x^2-5x} \times \frac{x^2-6x+5}{x^2-3x} = \frac{(x-3)(x-2)(x-5)(x-1)}{x^2(x-5)(x-3)}$$

$$= \frac{(x-1)(x-2)}{x^2}.$$

9. $$\frac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}} = \frac{x^2-6x-(x-6)+6}{x-6} \div \frac{x^2-6x-(2x-12)+3}{x-6}$$

$$= \frac{x^2-7x+12}{x-6} \times \frac{x-6}{x^2-8x+15} = \frac{(x-4)(x-3)(x-6)}{(x-6)(x-5)(x-3)} = \frac{x-4}{x-5}.$$

10. $$1+\frac{x}{1+x+\dfrac{2x^2}{1-x}} = 1+\frac{x}{\dfrac{1+x^2}{1-x}} = 1+\left(x \times \frac{1-x}{1+x^2}\right) = 1+\frac{x-x^2}{1+x^2}$$

$$\frac{1+x}{1+x^2}.$$

11. $$\frac{1+\dfrac{a-1}{a+1}}{1-\dfrac{a-1}{a+1}} = \frac{2a}{a+1} \div \frac{2}{a+1} = \frac{2a}{a+1} \times \frac{a+1}{2} = \frac{2a(a+1)}{2(a+1)} = a.$$

12. $$\frac{1-\dfrac{2pq}{p^2+q^2}}{\dfrac{p^3-q^3}{p-q}-3pq} = \frac{p^2+q^2-2pq}{p^2+q^2} \div (p^2+pq+q^2-3pq)$$

$$= \frac{p^2-2pq+q^2}{p^2+q^2} \times \frac{1}{p^2-2pq+q^2} = \frac{1}{p^2+q^2}.$$

13. $$\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}} = \frac{2a^2+2b^2}{a^2-b^2} \div \frac{4ab}{a^2-b^2} = \frac{2(a^2+b^2)}{a^2-b^2} \times \frac{a^2-b^2}{4ab} = \frac{a^2+b^2}{2ab}.$$

14. $$\frac{\dfrac{3x}{2}+\dfrac{x-1}{3}}{\dfrac{13}{6}(x+1)-\dfrac{x}{3}-\dfrac{5}{2}} = \frac{11x-2}{6} \div \frac{11x-2}{6} = \frac{11x-2}{6} \times \frac{6}{11x-2} = 1.$$

15. $$\frac{\dfrac{2x}{x+y}+\dfrac{y}{x-y}-\dfrac{y^2}{x^2-y^2}}{\dfrac{1}{x+y}+\dfrac{x}{x^2-y^2}}=\frac{2x^2-xy}{x^2-y^2}\div\frac{2x-y}{x^2-y^2}=\frac{x\,(2x-y)}{x^2-y^2}\times\frac{x^2-y^2}{2x-y}$$
$$=x.$$

16. $$\frac{\dfrac{x}{x-y}-\dfrac{y}{x+y}}{\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{x^2-y^2}}=\frac{x^2+y^2}{x^2-y^2}\div\frac{x^4+y^4}{x^4-y^4}=\frac{x^2+y^2}{x^2-y^2}\times\frac{x^4-y^4}{x^4+y^4}$$
$$=\frac{(x^2+y^2)\,(x^2+y^2)\,(x^2-y^2)}{(x^2-y^2)\,(x^4+y^4)}=\frac{(x^2+y^2)^2}{x^4+y^4}.$$

17. $$\frac{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}{\dfrac{a^2-b^2}{a^2+b^2}+\dfrac{a^2+b^2}{a^2-b^2}}=\frac{2a^2+2b^2}{a^2-b^2}\div\frac{2a^4+2b^4}{a^4-b^4}=\frac{2\,(a^2+b^2)}{a^2-b^2}\times\frac{a^4-b^4}{2\,(a^4+b^4)}$$
$$=\frac{2\,(a^2+b^2)\,(a^2+b^2)\,(a^2-b^2)}{2\,(a^2-b^2)\,(a^4+b^4)}=\frac{(a^2+b^2)^2}{a^4+b^4}.$$

Art. 191.

9. $$3\,(x-5)-5\,(x-3)=21x-46,$$
$$3x-15-5x+15=21x-46,$$
$$-23x=-46,$$
$$x=2.$$

10. $$9\,(2x-1)-7x=89-3\,(4x+2),$$
$$18x-9-7x=89-12x-6,$$
$$23x=92,$$
$$x=4.$$

11. $$7\,(x-10)+54x=2\,(1-x)-3\,(5-2x),$$
$$7x-70+54x=2-2x-15+6x,$$
$$57x=57,$$
$$x=1.$$

12. $$\frac{x}{3}+\frac{2x}{5}=3-\frac{x}{6},$$
$$10x+12x=90-5x,$$
$$27x=90,$$
$$x=3\tfrac{1}{3}.$$

13. $$\frac{6x}{7}-\frac{5x}{4}+31+\frac{3x}{2}=0,$$
$$24x-35x+868+42x=0,$$
$$31x=-868,$$
$$x=-28.$$

14. $$19x-4x = \frac{35}{2} - \frac{7x-2}{2},$$
$$38x-8x = 35-7x+2,$$
$$37x = 37,$$
$$x = 1.$$

15. $$x + \frac{3x-5}{2} = 12 - \frac{2x-4}{3},$$
$$6x+9x-15 = 72-4x+8,$$
$$19x = 95,$$
$$x = 5.$$

16. $$\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} - 17 = x,$$
$$30x+20x+15x+12x-1020 = 60x,$$
$$17x = 1020,$$
$$x = 60.$$

17. $$\frac{x+1}{3} - \frac{x+4}{5} = 16 - \frac{x+3}{4},$$
$$20(x+1)-12(x+4) = 960-15(x+3),$$
$$20x+20-12x-48 = 960-15x-45,$$
$$23x = 943,$$
$$x = 41.$$

18. $$3x - \frac{11x-37}{2} = 5 - \frac{2x+6}{5},$$
$$30x-5(11x-37) = 50-2(2x+6),$$
$$30x-55x+185 = 50-4x-12,$$
$$-21x = -147,$$
$$x = 7.$$

19. $$\frac{x-1}{7} = 7 - \frac{23-x}{5} - \frac{4+x}{4},$$
$$20(x-1) = 980-28(23-x)-35(4+x),$$
$$20x-20 = 980-644+28x-140-35x,$$
$$27x = 216,$$
$$x = 8.$$

20. $$7x + \frac{55}{4} - \frac{x}{2} = \frac{4x}{5} - \frac{35}{4} + \frac{41x}{5},$$
$$140x + 275 - 10x = 16x - 175 + 164x,$$
$$-50x = -450,$$
$$x = 9.$$

21. $$\frac{x-3}{4} - \frac{2x-5}{6} - \frac{41}{60} = \frac{3x-8}{5} - \frac{5x+6}{15},$$
$$15x - 45 - 20x + 50 - 41 = 36x - 96 - 20x - 24,$$
$$-21x = -84,$$
$$x = 4.$$

22. $$\frac{7x+9}{4} - \left(x - \frac{2x-1}{9}\right) = 7,$$
$$63x + 81 - 36x + 8x - 4 = 252,$$
$$35x = 175,$$
$$x = 5.$$

23. $$\frac{7+9x}{4} - \left(1 - \frac{2-x}{9}\right) = 7x,$$
$$63 + 81x - 36 + 8 - 4x = 252x,$$
$$-175x = -35,$$
$$x = \tfrac{1}{5}.$$

24. $$\frac{3x-11}{4} - \frac{28-9x}{8} = 4x - 14\tfrac{3}{4},$$
$$6x - 22 - 28 + 9x = 32x - 118,$$
$$-17x = -68,$$
$$x = 4.$$

25. $$2\left(\frac{x-8}{3}\right) - 3\left(\frac{9-x}{4}\right) - 5\left(\frac{x-11}{6}\right) = 7 - 3\left(\frac{x-17}{8}\right),$$
$$16x - 128 - 162 + 18x - 20x + 220 = 168 - 9x + 153,$$
$$23x = 391,$$
$$x = 17.$$

26. $$\frac{3x-1}{2x-1}-\frac{4x-2}{3x-2}=\frac{1}{6},$$

$$54x^2-54x+12-48x^2+48x-12 = 6x^2-7x+2,$$

$$x = 2.$$

27. $$\frac{3+x}{3-x}-\frac{2+x}{2-x}-\frac{1+x}{1-x}=1,$$

$$6-7x+x^2-6+5x+2x^2-x^3-6-x+4x^2-x^3 = 6-11x+6x^2-x^3,$$

$$8x = 12,$$

$$x = 1\tfrac{1}{2}.$$

28. $$\frac{2}{2x-3}=\frac{6}{3x+2}-\frac{1}{x-2},$$

$$6x^2-8x-8 = 12x^2-42x+36-6x^2+5x+6,$$

$$29x = 50,$$

$$x = 1\tfrac{21}{29}.$$

29. $$5x-[8x-3\{16-6x-(4-5x)\}] = 6,$$

$$5x-8x+48-18x-12+15x = 6,$$

$$6x = -30,$$

$$x = 5.$$

30. $$(x-5)(x-2)-(x-5)(2x-5)+(x+7)(x-2) = 0,$$

$$x^2-7x+10-2x^2+15x-25+x^2+5x-14 = 0,$$

$$13x = 29,$$

$$x = 2\tfrac{3}{13}.$$

Art. 193.

1. $$\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}.$$

Mult. by 36, $$9x+20=\frac{144x-432}{5x-4}+9x.$$

Omitting $9x$ and reducing, $20 = \dfrac{144x-432}{5x-4}$.

Whence, $100x-80 = 144x-432$,

and $x = 8$.

2. $$\frac{7x+16}{21} - \frac{x+8}{4x-11} = \frac{x}{3}.$$

Mult. by 21, $7x+16 - \dfrac{21x+168}{4x-11} = 7x.$

Omitting $7x$ and reducing,

$$64x-176-21x-168 = 0.$$

Whence, $x = 8$.

3. $$\frac{6x+7}{9} + \frac{7x-13}{6x+3} = \frac{2x+4}{3}.$$

Mult. by 9, $6x+7 + \dfrac{63x-117}{6x+3} = 6x+12.$

Omitting $6x$ and reducing, $63x-117 = 5(6x+3)$.

Whence, $x = 4$.

4. $$\frac{4x+3}{9} = \frac{8x+19}{18} - \frac{7x-29}{5x-12}$$

Mult. by 18, $8x+6 = 8x+19 - \dfrac{126x-522}{5x-12}.$

Omitting $8x$, $0 = 13(5x-12)-126x+522.$

Whence, $x = 6$.

5. $$\frac{3x-2}{2x-5} - \frac{21-3x}{5} = \frac{6x+13}{10}$$

Mult. by 10, $\dfrac{30x-20}{2x-5} - 42+6x = 6x+13.$

Omitting $6x$, $30x-20-55(2x-5) = 0.$

Whence, $x = 3\frac{3}{16}$.

6. $$\frac{6x+31}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}.$$

Mult. by 15, $$6x+31-\frac{45x+75}{5x-25}=6x.$$

Omitting $6x$, $$31(5x-25)-45x-75=0.$$

Whence, $$x=7\tfrac{8}{11}.$$

7. $$\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}.$$

Mult. by x^2+5x+6, $$4x+12+7x+14=37.$$

Whence, $$x=1.$$

8. $$\frac{25-\dfrac{x}{3}}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=5+\frac{23}{x+1}.$$

Uniting first fraction with second member of equation,

$$\frac{16x+4\frac{1}{5}}{3x+2}=\frac{5\frac{1}{3}x+3}{x+1}.$$

Whence, $$16x^2+19\tfrac{2}{3}x+6=16x^2+20\tfrac{1}{5}x+4\tfrac{1}{5}.$$

Hence, $$\frac{8x}{15}=\frac{27}{15},$$

and $$x=3\tfrac{3}{8}.$$

9. $$\frac{1}{3}-\frac{7x-1}{6\frac{1}{2}-3x}=\frac{8(x-\frac{1}{2})}{3(x-2)}.$$

Uniting the terms of left member,

$$\frac{19-48x}{39-18x}=\frac{8x-4}{3x-6}.$$

Clearing of fractions,

$$345x-144x^2-114=384x-144x^2-156.$$

Whence, $$x=1\tfrac{1}{13}.$$

10. $$\frac{9x+5}{14}+\frac{8x-7}{6x+2}-\frac{36x+15}{56}=\frac{10\frac{1}{4}}{14}.$$

Uniting 1st, 3d, and 4th terms,

$$\frac{8x-7}{6x+2}=\frac{9}{14}.$$

Clearing of fractions, $112x-98 = 54x+18.$

Whence, $x = 2.$

11.
$$\frac{x-7}{x+7}-\frac{2x-15}{2x-6}+\frac{1}{2(x+7)} = 0.$$

Uniting 1st and 3d terms, and transposing 2d term,

$$\frac{2x-13}{2x+14} = \frac{2x-15}{2x-6}.$$

Clearing of fractions,

$$4x^2-33x+78 = 4x^2-2x-210.$$

Whence, $x = 8.$

12.
$$\frac{6x-7\frac{1}{3}}{13-12x}+2x+\frac{1+16x}{24} = 4\tfrac{5}{18}-\frac{12\frac{5}{8}-8x}{8}.$$

Uniting all terms but the first,

$$\frac{6x-7\frac{1}{3}}{13-12x} = \frac{1}{6}.$$

Clearing of fractions, $36x-44 = 13-12x.$

Whence, $x = 1\tfrac{3}{16}.$

13.
$$\frac{18x-19}{28}+\frac{11x+21}{6x+14} = \frac{9x+15}{14}.$$

Transposing 1st term and uniting,

$$\frac{11x+21}{6x+14} = \frac{7}{4}.$$

Clearing of fractions, $44x+84 = 42x+98.$

Whence, $x = 7.$

14.
$$\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3} = \frac{7x}{12}-\frac{x+16}{36}.$$

Uniting all terms but the 2d, and transposing 2d,

$$\frac{25}{18} = \frac{13x-2}{17x-32}.$$

Clearing of fractions, $425x - 800 = 234x - 36$.

Whence, $x = 4$.

15. $$\frac{10x+17}{18} - \frac{12x+2}{13x-16} = \frac{5x-4}{9}.$$

Uniting 1st and 3d terms, and transposing 2d,

$$\frac{25}{18} = \frac{12x+2}{13x-16}.$$

Clearing of fractions,

$$325x - 400 = 216x + 36.$$

Whence, $x = 4$.

16. $$\frac{5}{1-5x} + \frac{4}{2x-1} = \frac{3}{3x-1}.$$

Clearing of fractions,

$$30x^2 - 25x + 5 - 60x^2 + 32x - 4 = -30x^2 + 21x - 3.$$

Whence, $x = \frac{2}{7}$.

17. $$\frac{6-5x}{15} - \frac{7-2x^2}{14(x-1)} = \frac{1+3x}{21} - \frac{2x-2\frac{1}{5}}{6} + \frac{1}{105}.$$

Uniting all terms but 2d, and transposing 2d,

$$\frac{-1-6x}{42} = \frac{7-2x^2}{14x-14}.$$

Clearing of fractions,

$$70x - 84x^2 + 14 = 294 - 84x^2.$$

Whence, $x = 4$.

18. $$\frac{1}{x-1} - \frac{1}{2(x+1)} - \frac{x+3}{2(x^2+1)} = \frac{6}{x^4-1}.$$

Mult. by $2(x^4-1)$,

$$2x^3 + 2x^2 + 2x + 2 - (x^3 - x^2 + x - 1) - (x^3 + 3x^2 - x - 3) = 12.$$

Whence, $x = 3$.

Art. 195.

1. $$x-a = (b-a)x.$$
$$x-a = bx-ax,$$
$$x-bx+ax = a,$$
$$x = \frac{a}{1-b+a}.$$

2. $$\frac{x}{a} - \frac{x-b}{c} = \frac{de}{ac}.$$
$$cx-ax+ab = de,$$
$$cx-ax = de-ab.$$
$$x = \frac{de-ab}{c-a}.$$

3. $$\frac{x}{a-b} - \frac{x}{a+b} = 1.$$
$$ax+bx-ax+bx = a^2-b^2,$$
$$2bx = a^2-b^2,$$
$$x = \frac{a^2-b^2}{2b}.$$

4. $$\frac{x}{a} + c = \frac{x}{b} - d.$$
$$bx+abc = ax-abd,$$
$$bx-ax = -abc-abd,$$
or, $$ax-bx = abc+abd,$$
$$x = \frac{ab(c+d)}{a-b}.$$

5. $$ax-c = d-bx.$$
$$ax+bx = d+c,$$
$$x = \frac{d+c}{a+b}.$$

6. $$3ax-4ab = 2ax-6ac.$$
$$ax = 4ab-6ac,$$
$$x = \frac{4ab-6ac}{a}$$
$$= 4b-6c.$$

7. $$ax^2+abx = cdx.$$
Dividing by x,
$$ax+ab = cd,$$
$$ax = cd-ab,$$
$$x = \frac{cd-ab}{a}.$$

8. $$ax^2+bx = dx^2+cx.$$
$$ax+b = dx+c,$$
$$ax-dx = c-b,$$
$$x = \frac{c-b}{a-d}.$$

9. $$\frac{a}{bx} + \frac{b}{ax} = a^2+b^2.$$
$$a^2+b^2 = abx(a^2+b^2).$$
Dividing by a^2+b^2,
$$abx = 1,$$
$$x = \frac{1}{ab}.$$

10. $$\frac{a(a-x)}{b} - \frac{b(b+x)}{a} = x.$$
$$a^3-a^2x-b^3-b^2x = abx,$$
$$a^2x+b^2x+abx = a^3-b^3,$$
$$x = \frac{a^3-b^3}{a^2+ab+b^2} = a-b.$$

11. $$\frac{a(x-a)}{b}+\frac{b(x-b)}{a}=x.$$

$$a^2x-a^3+b^2x-b^3=abx,$$

$$a^2x+b^2x-abx=a^3+b^3,$$

$$x=\frac{a^3+b^3}{a^2-ab+b^2}=a+b.$$

12. $$a(x+a)-b(x-b)=3ax+(a-b)^2.$$

$$ax+a^2-bx+b^2=3ax+a^2-2ab+b^2.$$

$$-2ax-bx=-2ab,$$

$$x=\frac{2ab}{2a+b}.$$

13. $$\frac{a}{x}+\frac{b}{a}=\frac{1}{3x}.$$

$$3a^2+3bx=a,$$

$$3bx=a-3a^2,$$

$$x=\frac{a(1-3a)}{3b}.$$

14. $$x+\frac{ax}{b}+\frac{cx}{b}=m.$$

$$bx+ax+cx=bm,$$

$$x=\frac{bm}{a+b+c}.$$

15. $$x(x-a)+x(x-b)=2(x-a)(x-b).$$

$$x^2-ax+x^2-bx=2x^2-2ax-2bx+2ab,$$

$$ax+bx=2ab,$$

$$x=\frac{2ab}{a+b}.$$

16. $$(x-a)(x-b)=(x-a-b)^2.$$

$$x^2-ax-bx+ab=x^2-2ax-2bx+a^2+2ab+b^2,$$

$$ax+bx=a^2+ab+b^2,$$

$$x=\frac{a^2+ab+b^2}{a+b}.$$

17. $$\frac{1}{a(b-x)}+\frac{1}{b(c-x)}=\frac{1}{a(c-x)}.$$

$$b(c-x)+a(b-x)=b(b-x),$$

$$bc-bx+ab-ax=b^2-bx,$$

$$ax=ab+bc-b^2,$$

$$x=\frac{b(a+c-b)}{a}.$$

18. $$\frac{x-a}{a-b} - \frac{x+a}{a+b} = \frac{2ax}{a^2-b^2}.$$

$$(a+b)(x-a)-(x+a)(a-b) = 2ax,$$
$$ax+bx-a^2-ab-ax-a^2+bx+ab = 2ax,$$
$$2bx-2ax = 2a^2,$$
$$x = \frac{a^2}{b-a}.$$

19. $$(a+x)(b+x)-a(b+c) = \frac{a^2c}{b} + x^2.$$

$$ab^2+b^2x+abx+bx^2-ab^2-abc = a^2c + bx^2,$$
$$b^2x+abx = a^2c+abc,$$
$$x = \frac{ac(a+b)}{b(a+b)} = \frac{ac}{b}.$$

20. $$\frac{2x+a}{b} - \frac{x-b}{a} = \frac{3ax+(a-b)^2}{ab}.$$

$$2ax+a^2-bx+b^2 = 3ax+a^2-2ab+b^2,$$
$$-ax-bx = -2ab,$$
$$x = \frac{2ab}{a+b}.$$

21. $$\frac{a^2+x^2}{ax} + \frac{a(x-a)}{x(x+a)} + \frac{ax}{x^2-a^2} + 2 = \frac{x(x+a)}{a(x-a)}.$$

$$(a^2+x^2)(x^2-a^2)+a^2(x^2-2ax+a^2)+a^2x^2+2ax(x^2-a^2)=x^2(x^2+2ax+a^2),$$
$$x^4-a^4+a^2x^2-2a^3x+a^4+a^2x^2+2ax^3-2a^3x = x^4+2ax^3+a^2x^2,$$
$$a^2x^2-4a^3x = 0,$$

or, $$a^2x^2 = 4a^3x,$$

whence, $$x = 4a.$$

22. $$\frac{1}{2}\left(x-\frac{a}{3}\right) - \frac{1}{3}\left(x-\frac{a}{4}\right) + \frac{1}{4}\left(x-\frac{a}{5}\right) = 0.$$

$$\frac{3x-a}{6} - \frac{4x-a}{12} + \frac{5x-a}{20} = 0,$$
$$30x-10a-20x+5a+15x-3a = 0,$$
$$25x = 8a,$$
$$x = \frac{8a}{25}.$$

23.

$$(a-b)(x-c)-(b-c)(x-a)=(c-a)(x-b).$$
$$ax-bx-ac+bc-bx+cx+ab-ac=cx-ax-bc+ab,$$
$$2ax-2bx=2ac-2bc,$$
$$x=\frac{ac-bc}{a-b}=c.$$

Art. 196.

1.

$$\frac{x+7}{3}+\frac{2x+6}{14}=\frac{6x-10}{7}.$$
$$14x+98+6x+18=36x-60,$$
$$-16x=-176,$$
whence, $x=11.$

2.

$$\frac{x}{7}+\frac{x}{3}=\frac{13x+14}{28}.$$
$$12x+28x=39x+42,$$
whence, $x=42.$

3.

$$\frac{3x+6}{13}=\frac{7x+13}{30}.$$
$$90x+180=91x+169,$$
whence, $x=11.$

4.

$$2+\frac{x-7}{3}=\frac{3x+3}{10}.$$
$$60+10x-70=9x+9,$$
whence, $x=19.$

5.

$$7-\left(\frac{x-7}{12}+\frac{7x-1}{20}\right)=\frac{5x+7}{18}-2.$$
$$1260-15x+105-63x+9=50x+70-360,$$
$$-128x=-1664,$$
whence, $x=13.$

6.

$$\frac{9-x}{3}+\frac{3x+5}{10}=\frac{7x-5}{25}-1.$$
$$450-50x+45x+75=42x-30-150,$$
$$-47x=-705,$$
whence, $x=15.$

7.

$$\frac{11(x-1)}{4}+\frac{11-x}{3}=\frac{9(x+3)}{12}+\frac{7(x+1)}{6}.$$
$$33x-33+44-4x=9x+27+14x+14,$$
$$6x=30,$$
whence, $x=5.$

8. $$\frac{1}{4}(x-7)+\frac{2}{5}(x+8)=\frac{2}{7}(2x+9)-\frac{1}{12}(x-15).$$

$$105x-735+168x+504=240x+1080-35x+525,$$

$$68x=1836,$$

whence, $$x=27.$$

9. $$5x+\frac{3x-1}{7}=\frac{5}{9}(6x+15)+\frac{1}{7}(3x-1).$$

Canceling $\frac{3x-1}{7}$ $$45x=30x+75,$$

$$15x=75,$$

whence, $$x=5.$$

10. $$\frac{7-x}{5}+\frac{3x-5}{7}+\frac{7x-3}{8}+\frac{14-4x}{18}=-1.$$

$$5096-728x+1560x-2600+3185x-1365+3920-1120x=-3640,$$

$$2897x=-8691,$$

whence, $$x=-3.$$

11. $$\frac{x-7}{10}-\frac{3x-1}{35}-\frac{4x+2}{45}=1.$$

$$63x-441-54x+18-56x-28=630,$$

$$-47x=1081,$$

whence, $$x=-23.$$

12. $$\frac{(3x+5)(5x+8)}{23}-\left(3x+\frac{x+3}{7}\right)=\frac{10x(3x-2)}{46}-6.$$

$$210x^2+476x+210-966x-46x-138=210x^2-140x-1932,$$

$$-396x=-2004,$$

whence, $$x=5\tfrac{2}{33}.$$

13. $$\frac{1}{2}(1+25x)-\left(1-\frac{4-2x}{9}\right)=12x.$$

$$9+225x-18+8-4x=216x,$$

$$5x=1,$$

whence, $$x=\tfrac{1}{5}.$$

14. $$\frac{1}{4}(9x-1)-\left(2x-\frac{3x-7}{2}\right)=5.$$

$$9x-1-8x+6x-14=20,$$

$$7x=35,$$

whence, $$x=5.$$

15. $$\frac{1}{x-4}-\frac{1}{x-3}=\frac{1}{x-2}-\frac{1}{x-1}.$$

$$\frac{1}{(x-4)(x-3)}=\frac{1}{(x-2)(x-1)},$$

$$x^2-3x+2=x^2-7x+12,$$

$$4x=10,$$

whence, $$x=2\tfrac{1}{2}.$$

16. $$\frac{x-3}{2}-\frac{x}{2}+16\tfrac{2}{3}=\frac{x}{9}+\frac{4x-17\frac{1}{2}}{29}.$$

$$261x-783-261x+8700=58x+72x-315,$$

$$-130x=-8232,$$

whence, $$x=63\tfrac{21}{65}.$$

17. $$(2-x)^2+8=[2(1+x)-x]\,x.$$

$$4-4x+x^2+8=2x+2x^2-x^2.$$

$$-6x=-12,$$

whence, $$x=2.$$

18. $$\frac{5x+3}{3}-\left(12+\frac{4x+9}{7}\right)+13=4.$$

$$35x+21-252-12x-27+273=84,$$

$$23x=69,$$

whence, $$x=3.$$

19. $$5.6x-\frac{.48x-3.7}{.5}-.8x=4x+5.8.$$

$$2.8x-.48x+3.7-.4x=2x+2.9,$$

$$-.08x=-.8,$$

whence, $$x=10.$$

20. $$\frac{2}{ac}+\frac{2}{ab}+\frac{4}{bc}=2ax+bx+cx.$$

$$2(2a+b+c)=abc(2a+b+c)x,$$

whence, $$x=\frac{2}{abc}.$$

21. $$\frac{b(c^2+x^2)}{cx}=bd+\frac{bx}{c}.$$

$$bc^2+bx^2=bcdx+bx^2,$$

whence, $$x=\frac{bc^2}{bcd}=\frac{c}{d}.$$

22. $$\frac{7x+10}{2x-10}-\frac{4x-29}{57}=\frac{x+45}{19}-\frac{7x-122}{57}.$$

$$\frac{7x+10}{2x-10}=\frac{x+45}{19}-\frac{7x-122}{57}+\frac{4x-29}{57},$$

$$\frac{7x+10}{2x-10}=\frac{3x+135-7x+122+4x-29}{57},$$

$$\frac{7x+10}{2x-10}=4,$$

$$7x+10=8x-40,$$

whence, $$x=50.$$

23. $$\frac{x-3}{x-5}-\frac{x+6}{x+4}=\frac{5x-2}{x^2-x-20}.$$

$$(x-3)(x+4)-(x+6)(x-5)=5x-2,$$

$$x^2+x-12-x^2-x+30=5x-2,$$

$$5x=20,$$

whence, $$x=4.$$

24. $$(2x+3)(3x+2)+(5x+7)(3-x)=(7x+27)(x-7)-2(3x-7)(x-11).$$

$$6x^2+13x+6+8x-5x^2+21=7x^2-22x-189-6x^2+80x-154,$$

$$-37x=-370,$$

whence, $$x=10.$$

25. $$7x - \left(10x - \frac{9x+5}{2x}\right) = 5-3x.$$

$$14x^2-20x^2+9x+5 = 10x-6x^2,$$

whence, $$x = 5.$$

26. $$\frac{a-x}{x-b} + \frac{x+c}{b+x} = \frac{a+x}{b-x} + \frac{x-c}{b+x},$$

or, $$\frac{x-a}{b-x} - \frac{a+x}{b-x} = \frac{x-c}{b+x} - \frac{x+c}{b+x}.$$

$$\frac{2a}{b-x} = \frac{2c}{b+x},$$

$$ab+ax = bc-cx,$$

$$ax+cx = bc-ab,$$

whence, $$x = \frac{b(c-a)}{a+c}.$$

27. $$(x+3)^2 - \frac{8x(x+5)}{10} = 7x - \left\{3x - \frac{x(x-5\frac{1}{2})}{5}\right\}.$$

$$10x^2+60x+90-8x^2-40x = 70x-30x+2x^2-11x,$$

$$-9x = -90,$$

whence, $$x = 10.$$

28. $$\frac{1}{amn+mnx} + \frac{1}{abc+bcx} = \frac{1}{a^2mc-mcx^2}.$$

$$\frac{1}{mn(a+x)} + \frac{1}{bc(a+x)} = \frac{1}{mc(a^2-x^2)},$$

$$bc(a-x)+mn(a-x) = bn,$$

$$abc-bcx+amn-mnx = bn,$$

$$bcx+mnx = abc+amn-bn,$$

whence, $$x = \frac{abc+amn-bn}{bc+mn}.$$

29.
$$\frac{a-bx}{b}-\frac{b-ax}{a}=ax+bx.$$
$$a^2-abx-b^2+abx=a^2bx+ab^2x,$$
$$a^2bx+ab^2x=a^2-b^2,$$
whence,
$$x=\frac{a^2-b^2}{a^2b+ab^2}=\frac{a-b}{ab}.$$

30.
$$\frac{ax}{b}-\frac{bx}{a}=a+b.$$
$$a^2x-b^2x=a^2b+ab^2,$$
whence,
$$x=\frac{ab(a+b)}{a^2-b^2}=\frac{ab}{a-b}.$$

31.
$$(2a+b)(x+1)=(2b+a)(x-1).$$
$$2ax+2a+bx+b=2bx+ax-2b-a,$$
$$ax-bx=-3a-3b,$$
whence,
$$x=\frac{3(a+b)}{b-a}.$$

32.
$$(a-b+c)(x-a)=(b-a+c)(x+a).$$
$$ax-bx+cx-a^2+ab-ac=bx-ax+cx+ab-a^2+ac,$$
$$2ax-2bx=2ac,$$
whence,
$$x=\frac{ac}{a-b}.$$

33.
$$x+\frac{x-4}{17}=3x-\frac{5-x}{34}-6.$$
$$34x+2x-8=102x-5+x-204,$$
$$-67x=-201,$$
whence,
$$x=3.$$

34.
$$\frac{4x-11}{8}+17x=\frac{7}{8}\left(19x-\frac{x-13}{7}\right).$$
$$28x-77+952x=924x+91,$$
$$56x=168,$$
whence,
$$x=3.$$

35. $$(a-x)(x-b)+(a+x)(x-b) = (a-b)^2.$$

$$ax-ab-x^2+bx+ax-ab+x^2-bx = a^2-2ab+b^2,$$

$$2ax = a^2+b^2,$$

whence, $$x = \frac{a^2+b^2}{2a}.$$

36. $$(x-7)(2-x)+(1\tfrac{1}{2}x-1)(\tfrac{2}{3}x+5) = 5(\tfrac{1}{6}x+7).$$

$$9x-14-x^2+x^2+6\tfrac{5}{6}x-5 = \tfrac{5}{6}x+35,$$

$$15x = 54,$$

whence, $$x = 3\tfrac{3}{5}.$$

37. $$(2x-3)^2 = \{11-(11\tfrac{3}{4}-x)\}\,4x.$$

$$4x^2-12x+9 = -3x+4x^2,$$

$$-9x = -9,$$

whence, $$x = 1.$$

38. $$\frac{(5x-7)(1-x)}{3x+2}+\frac{2\tfrac{1}{2}x+4}{1\tfrac{1}{2}} = 20.$$

$$18x-10\tfrac{1}{2}-\tfrac{15}{2}x^2+\tfrac{15}{2}x^2+17x+8 = 90x+60,$$

$$-55x = 62\tfrac{1}{2},$$

whence, $$x = -1\tfrac{3}{22}.$$

Art. 198.

1. Let $x =$ No. of dollars that the man had.

Then will $\frac{3}{10}x =$ cost of clothes,

and $\frac{1}{4}x =$ cost of watch.

Hence, $x-\frac{3}{10}x-\frac{1}{4}x = 54.$

Whence, $9x = 1080,$

and $x = 120$, No. of dollars that the man had.

2. Let $x =$ the number.

Then will $x-(\frac{1}{2}x+\frac{1}{3}x) = 36.$

Whence, $x = 216$, the number.

3. Let $x =$ No. of acres in the farm.

Then will $\frac{1}{3}x =$ No. of acres in corn,

$\frac{1}{4}x =$ No. of acres in wheat,

and $\frac{1}{6}x =$ No. of acres in oats.

Hence, $x = \frac{1}{3}x + \frac{1}{4}x + \frac{1}{6}x + 37\frac{1}{2}$.

Whence, $x = 150$, No. of acres in the farm.

4. Let $x =$ No. of dollars in the salary.

Then will $\frac{2}{9}x =$ No. of dollars spent on rent,

$\frac{1}{4}x =$ No. of dollars spent on food,

$\frac{1}{12}x =$ No. of dollars spent on taxes,

and $\frac{1}{6}x =$ No. of dollars spent on clothes.

Hence, $x = \frac{2}{9}x + \frac{1}{4}x + \frac{1}{12}x + \frac{1}{6}x + 500$.

Whence, $36x = 8x + 9x + 3x + 6x + 18000$,

or, $x = 1800$, No. of dollars in the salary.

5. Let $x =$ cost of coat.

Then will $\frac{2}{5}x =$ cost of hat.

Hence, $x + \frac{2}{5}x = \$21$.

Whence, $x = \$15$, cost of coat;

and $\frac{2}{5}x = \$6$, cost of hat.

6. Let $x =$ cost of lot.

Then will $x + \$1425 =$ cost of house.

Hence, $x + (x + \$1425) = \$3775$.

Whence, $x = \$1175$, cost of lot;

and $x + \$1425 = \$2600$, cost of house.

7. Let $x =$ No. of dollars that the father earned.

Then will $x - 3760 =$ No. of dollars that the son earned.

Hence, $x + (x - 37.60) = 165$.

Whence, $x = 101.30$, No. of dollars that the father earned;

and $x - 37.60 = 63.70$, No. of dollars that the son earned.

8. Let x = cost of horse.

Then will $x-\$25$ = cost of carriage,

and $x-\$65$ = cost of sleigh.

Hence, $x+(x-\$25)+(x-\$65) = \$600.$

Whence, $x = \$230$, cost of horse;

$x-\$25 = \$205$, cost of carriage;

and $x-\$65 = \$165$, cost of sleigh.

9. Let x = son's share of the property.

Then will $x-\$2500$ = daughter's share,

and $x+\$3300$ = wife's share.

Hence, $x+(x-\$2500)+(x+\$3200) = \$15800.$

Whence, $x = \$5000$, son's share;

$x-\$2500 = \$2500$, daughter's share;

and $x+\$3300 = \$8300$, wife's share.

10. Let x = No. of miles traveled on second day.

Then will $x-15$ = No. of miles traveled on first day,

and $x+20$ = No. of miles traveled on third day.

Hence, $x+(x-15)+(x+20) = 155.$

Whence, $x = 50$, No of miles traveled on 2d day;

$x-15 = 35$, No. of miles traveled on 1st day;

and $x+20 = 70$, No. of miles traveled on 3d day.

11. Let x = No. pounds in mixture worth \$.40.

Then will $100-x$ = No. pounds in mixture worth \$.50.

Hence, $\$.40x+\$.50(100-x) = \$44.50,$

or, $\$.10x = \$5.50.$

Whence, $x = 55$, No. of pounds worth \$.40;

and $100-x = 45$, No. of pounds worth \$.50.

12. Let $x =$ No. of days that they traveled.

Then will $15x+19x = 204.$

Whence, $x = 6$, No. of days.

13. Let $x =$ No. of days they worked.

Then will $\$5.00+5(\$3.75x)+3(\$1.35x) = \$338.60,$

or, $\$27.80x = \$333.60.$

Whence, $x = 12$, No. of days.

14. Let $x =$ age of first child.

Then will $x+3 =$ age of second child,

$x+6 =$ age of third child,

and $x+9 =$ age of fourth child.

Hence, $5[x+(x+3)+(x+6)+(x+9)] = 250.$

Whence, $x = 8$, age of first child;

$x+3 = 11$, age of second child;

$x+6 = 14$, age of third child;

and $x+9 = 17$, age of fourth child.

15. Let $x =$ cost of overcoat.

Then will $\$70-x =$ cost of clothes,

and $\$39-x =$ cost of boots.

Hence, $(\$70-x)+(\$39-x) = \$51.$

Whence, $x = \$29$, cost of overcoat;

$\$70-x = \$41$, cost of clothes;

and $\$39-x = \$10$, cost of boots.

16. Let $x =$ No. of dollars man can earn.

Then will $54-x =$ No. of dollars oldest son can earn,

and $48-x =$ No. of dollars youngest son can earn.

Hence, $(54-x)+(48-x) = 32.$

Whence, $x = 35$, No. of dollars man can earn;

$54-x = 19$, No. of dollars oldest son can earn;

and $48-x = 13$, No. of dollars youngest son can earn.

17. Let $x =$ cost of a bushel of wheat.
Then will $\$2.05-x =$ cost of a bushel of rye,
and $\$1.60-x =$ cost of a bushel of oats.
Hence, $(\$2.05-x)+(\$1.60-x) = \$1.39$.
Whence, $x = \$1.13$, cost of wheat;
$\$2.05-x = \$.92$, cost of rye;
and $\$1.60-x = \$.47$, cost of oats.

18. Let $x =$ B's present age.
Then will $2x =$ A's present age.
Hence, $(x-15)+(2x-15) = 84$.
Whence, $x = 38$, B's present age;
and $2x = 76$, A's present age.

19. Let $x =$ Mrs. B's present age.
Then will $1\frac{1}{5}x =$ Mr. B's present age.
Hence, $1\frac{1}{5}x-20 = 1\frac{2}{5}(x-20)$.
Whence, $x = 40$, Mrs. B's present age;
and $1\frac{1}{5}x = 48$, Mr. B's present age.

20. Let $3x =$ the first number.
Then will $4x =$ the second number.
Hence, $3x+50 : 4x+50 :: 5 : 6$.
From this, $18x+300 = 20x+200$.
Whence, $x = 25$;
$3x = 75$, first number;
and $4x = 100$, second number.

21. Let $5x =$ the price of the house.
Then will $3x =$ price of lot.
Hence, $5x-\$500 : 3x-\$500 :: 9 : 5$.
From this, $25x-\$2500 = 27x-\$4500$.
Whence, $x = \$1000$;
$5x = \$5000$, price of house;
and $3x = \$3000$, price of lot.

22. Let $x =$ value of one sheep.

Then will $\frac{1}{4}(94x-72x) = \$44.$

Whence, $x = \$4$, value of one sheep.

23. Let $x =$ No. of dollars that first person had.

Then will $2x =$ No. of dollars that second person had,

and $2x-25 =$ No. of dollars that third person had.

Hence, $x+2x+(2x-25) = 400.$

Whence, $x = 85$, No. of dollars that first had;

$2x = 170$, No. of dollars that second had;

and $2x-25 = 145$, No. of dollars that third had.

24. Let $x =$ No. of dollars that each invested.

Hence, $x+875 : x-1575 :: 3 : 1.$

Whence, $x+875 = 3x-4725,$

and $x = 2800$, No. of dollars that each invested.

25. Let $x =$ No. of sheep bought.

Then will $\frac{\$480}{x} =$ cost of one sheep,

and $\frac{\$108}{\frac{1}{4}(x-12)} =$ selling price of one sheep.

Hence, $\frac{\$480}{x} = \frac{\$108}{\frac{1}{4}(x-12)},$

or, $\$480x-\$5760 = \$432x.$

Whence, $x = 120$, No. of sheep bought.

26. Let $x =$ No. of children.

Then will $\$.25x-\$.25 = \$.20x+\$.80.$

Whence, $x = 21$, No. of children.

27. Let $x =$ No. of dollars spent for watch.

Then will $185-x =$ No. of dollars spent for chain.

Hence, $185-x+18 = \frac{2}{5}x.$

Whence, $x = 145$, No. of dollars spent for watch;

and $185-x = 40$, No. of dollars spent for chain.

28. Let $x =$ value of house.

Then will $\$12800-x =$ value of lot.

Hence, $7x = 9\,(\$12800-x)$.

Whence, $x = \$7200$, value of house;

and $\$12800-x = \$5600$, value of lot.

29. Let $x =$ value of cargo.

Then will $\$127400-x =$ value of steamer.

Hence, $\$127400-x+\$9100 = 2\frac{3}{4}x$.

Whence, $x = \$36400$, value of cargo;

and $\$127400-x = \$91000$, value of steamer.

30. Let $x =$ No. of dollars loaned at 5%.

Then will $5000-x =$ No. of dollars loaned at 6%.

Hence, $.05x+.06\,(5000-x) = 280$.

Whence, $x = 2000$, No. of dollars loaned at 5%

31. Let $x =$ No. cu. in. of copper used.

Then will $100-x =$ No. cu. in. of tin used.

Hence, $5\frac{1}{4}x+4\frac{1}{4}(100-x) = 505$.

Whence, $x = 80$, No. cu. in. of copper;

and $100-x = 20$, No. cu. in. of tin.

See Appendix.

32. Let $x =$ No. oz. in first piece.

Then will $x+12 =$ No. oz. in second piece,

and $x+21 =$ No. oz. in third piece.

Hence, $x+(x+12)+(x+21) = 48$.

Whence, $x = 5$, No. oz. in first;

$x+12 = 17$. No. oz. in second;

and $x+21 = 26$. No. oz. in third.

33. Let $x =$ the number.

Then will $(4x-26)-150 = 200-(7x-20)$.

Whence, $x = 36$, the number.

34. Let $x =$ No. of dollars A will have.

Then will $5\frac{1}{2}+x =$ No. of dollars B will have,

and $\frac{2}{3}[x+(5\frac{1}{2}+x)] =$ No. of dollars C will have.

Hence, $x+(5\frac{1}{2}+x)+\frac{2}{3}[x+(5\frac{1}{2}+x)] = 155$.

Whence, $x = 43.75$, No. dollars A will have;

$5\frac{1}{2}+x = 49.25$, No. dollars B will have;

and $\frac{2}{3}[x+(5\frac{1}{2}+x)] = 62$, No. dollars C will have.

35. Let $x =$ height of the house.

Then will $\frac{5}{7}x-8 =$ height above window,

and $\frac{3}{4}\frac{1}{2}x-8 =$ height below window.

Hence, $(\frac{5}{7}x-8)+(\frac{3}{4}\frac{1}{2}x-8)+8 = x$.

Whence, $x = 48$, height of house.

36. Let $x =$ No. of sheep A had at first.

Then will $x-11 =$ No. sheep B had at first.

Hence, $x-17 = \dfrac{(x-11)+17}{2}$.

Whence, $x = 40$, No. sheep A had at first;

and $x-11 = 29$, No. sheep B had at first.

37. Let $x =$ first part.

Then will $96-x =$ second part.

Hence, $\dfrac{x}{5} = \dfrac{96-x}{3}$.

Whence, $x = 60$, the first part;

and $96-x = 36$, the second part.

38. Let x = No. of dollars C should receive.

Then will $\frac{2}{3}x$ = No. dollars B should receive,

and $2\frac{1}{2}(\frac{2}{3}x)$ = No. dollars A should receive.

Hence, $x+\frac{2}{3}x+2\frac{1}{2}(\frac{2}{3}x) = 100.$

Whence, $x = 30$, No. dollars C should receive;

$\frac{2}{3}x = 20$, No. dollars B should receive;

and $2\frac{1}{2}(\frac{2}{3}x) = 50$, No. dollars A should receive.

39. Let x = what father earns.

Then will $x-\$.35$ = what son earns.

Hence, $11x-6(x-\$.35) = \$12.10.$

Whence, $x = \$2.00$, what father earns;

and $x-\$.35 = \$1.65$, what son earns.

40. Let x = A's fortune.

Then will $\frac{x}{3}$ = B's fortune.

Hence, $x-\frac{x}{2}-\$250 = \frac{x}{3}-\frac{x}{6}+\$250.$

Whence, $x = \$1500$, A's fortune;

and $\frac{x}{3} = \$500$, B's fortune.

41. Let x = first number.

Then will $97-x$ = second number.

Hence, $x-(97-x) = 17.$

Whence, $x = 57$, first number;

and $97-x = 40$, second number.

42. Let x = No. of dollars divided.

Then will $10+\frac{1}{2}x+\frac{1}{4}x+5 = x.$

Whence, $x = 60$, No. of dollars divided.

43. Let $x =$ B's present age.

Then will $x+10 =$ A's present age.

Hence, $x+20 = 2(x-5)$.

Whence, $x = 30$, B's present age;

and $x+10 = 40$, A's present age.

44. Let $x =$ No. of dollars invested.

Then will $.08(\frac{1}{8}x)+.04\frac{1}{2}(\frac{1}{4}x)+.07\frac{1}{2}(\frac{1}{6}x)+.05(\frac{11}{24}x) = 680$,

or, $.24x+.27x+.30x+.55x = 16320$.

Whence, $x = 12000$, No dollars.

45. Let $x =$ No. of \$.10 pieces.

Then will $70-x =$ No. of \$.25 pieces.

Hence, $\$.10x+\$.25(70-x) = \$10.00$.

Whence, $x = 50$, No. \$.10 pieces;

and $70-x = 20$, No. \$.25 pieces.

46. Let $x =$ second number.

Then will $1\frac{1}{4}x =$ first number,

and $\frac{2}{3}(x+1\frac{1}{4}x)+17 =$ third number.

Hence, $x+1\frac{1}{4}x+[\frac{2}{3}(x+1\frac{1}{4}x)+17] = 272$.

Whence, $x = 68$, second number;

$1\frac{1}{4}x = 85$, first number;

and $\frac{2}{3}(x+1\frac{1}{4}x)+17 = 119$, third number.

47. Let $x =$ No. of owners.

Then $x : x-5 :: 40 : 30$.

From this, $30x = 40x-200$.

Whence, $x = 20$, No. of owners.

48. Let $x =$ No. of dollars spent for carriage.

Then will $\frac{1}{2}x-10 =$ No. dollars spent for harness,

and $x+(\frac{1}{2}x-10)+10 =$ No. dollars spent for horse.

Hence, $x+(\frac{1}{2}x-10)+[x+(\frac{1}{2}x-10)+10] = 290.$

Whence, $x = 100$, No. dollars spent for carriage;

$\frac{1}{2}x-10 = 40$, No. dollars spent for harness;

and $x+(\frac{1}{2}x-10)+10 = 150$, No. dollars spent for horse.

49. Let $x =$ income.

Then will $\frac{3}{5}x+\$200 =$ expenditure.

Hence, $5(\frac{2}{5}x-\$200) = \frac{7}{10}x+\$300.$

Whence, $x = \$1000$, income;

and $\frac{3}{5}x+\$200 = \$800$, expenditure.

50. Let $x =$ No. days that he worked.

Then will $30-x =$ No. days that he was idle.

Hence, $\$5x-\$3.75(30-x) = \$10.$

Whence, $x = 14$, No. days that he worked;

and $30-x = 16$, No. days that he was idle.

51. Let $x =$ No. of lemons bought.

Then will $\$.05x+\$.04 =$ money first boy had,

and $\$.02(1\frac{2}{3}x)+\$.29 =$ money second boy had.

Hence, $\$.05x+\$.04 = \$.02(1\frac{2}{3}x)+\$.29.$

Whence, $x = 15$, No. of lemons bought;

$\$.05x+\$.04 = \$.79$, money first boy had;

and $\$.02(1\frac{2}{3}x)+\$.29 = \$.79$, money second boy had.

52. Let $x =$ No. of dollars invested.

Then will $.07(\frac{2}{3}x+500)-49 = .06(\frac{7}{8}x-700).$

Whence, $x = 4800$, No. of dollars invested.

53. Let $x =$ No. of pounds of \$.15 sugar.

Then will $100-x =$ No. of pounds of \$.11 sugar.

Hence, $\$.15x+\$.11(100-x) = \$.12\frac{2}{5}(100).$

Whence, $x = 35$, No. pounds of \$.15 sugar;

and $100-x = 65$, No. pounds of \$.11 sugar.

54. Let $x = $ No. of women.

Then will $x+4 = $ No. of men,

and $x+(x+4)+10 = $ No. of children.

Hence, $x+(x+4)+[x+(x+4)+10] = 90.$

Whence, $x = 18$, No. of women;

$x+4 = 22$, No. of men·

and $x+(x+4)+10 = 50$, No. of children.

55. Let $x = $ youngest son's share.

Then will $x+\$20 = $ second son's share,

$x+\$40 = $ third son's share,

$x+\$60 = $ fourth son's share,

and $x+\$80 = $ fifth son's share.

Hence, $5x+\$200 = \$1000.$

Whence, $x = \$160$, youngest son's share.

56. Let $\frac{3}{4}x = $ A's age at marriage.

Then will $\frac{2}{3}x = $ his wife's age at marriage.

Hence, $\frac{3}{4}x+12 : \frac{2}{3}x+12 :: 13 : 12.$

From this, $9x+144 = \frac{26}{3}x+156.$

Whence, $x = 36;$

$\frac{3}{4}x = 27$, A's age at marriage;

and $\frac{2}{3}x = 24$, his wife's age at marriage.

57. Let $x = $ No. of miles apart at starting.

Then will $\frac{1}{2}x+66 = $ No. of miles A traveled,

and $\frac{1}{2}x-66 = $ No. of miles B traveled.

Hence, $\frac{1}{2}x+66 : \frac{1}{2}x-66 :: \frac{11}{2} : \frac{22}{7}.$

From this, $44x+5808 = 77x-10164.$

Whence, $x = 484$, No. of miles apart.

58. Let $x =$ No. of officers in the company.

Then will $15x =$ No. of soldiers,

and $x+15x =$ total No. of men in the company.

Hence, $\frac{x+15x}{x-10} = 20.$

Whence, $x = 50$, No. of officers;

and $x+15x = 800$, total No. of men.

59. Let $x =$ No. of dollars in first part.

Then will $5000-x =$ No. of dollars in second part.

Hence, $6\,(.10)\,x + x = 5000-x + 5\,(.08)\,(5000-x),$

or, $1.60x = 7000-1.40x.$

Whence, $x = 2333\frac{1}{3}$, No. of dollars in first part;

and $5000-x = 2666\frac{2}{3}$, No. of dollars in second part.

60. $\$1030 =$ debt at end of six months.

Let $x =$ No. of bushels given.

Then will $\$.75x =$ worth of potatoes,

and $\$.50x =$ worth of apples.

Hence, $\$.75x + \$.50x = \$1030.$

Whence, $x = 824$, No. of bushels given.

61. Let $x =$ No. receiving \$.13.

Then will $16-x =$ No. receiving \$.11.

Hence, $\$.13x + \$.11\,(16-x) = \$1.90.$

Whence, $x = 7$, No. receiving \$.13.

62. Let $x =$ smaller sum.

Then will $\$2000-x =$ greater sum.

Hence, $9\,(.07x) = 7\,[.06\,(\$2000-x)].$

Whence, $x = \$800$, smaller sum;

and $\$2000-x = \$1200$, greater sum.

63. Let $x =$ No. of days he worked.

Then will $420-x =$ No. of days he was idle.

Hence, $\$.50x = \$.20\,(420-x)$.

Whence, $x = 120$, No. of days he worked.

64. Let $x =$ No. of \$.03 pieces.

Then will $39-x =$ No. of \$.05 pieces.

Hence, $\$.03x + \$.05\,(39-x) = \$1.51$.

Whence, $x = 22$, No. of \$.03 pieces;

and $39-x = 17$, No. of \$.05 pieces.

65. Let $x =$ length of upper piece.

Then will $\frac{2x+4}{3} =$ length of lower piece,

and $2\left(\frac{2x+4}{3}\right)+3 =$ length of middle piece.

Hence, $x+\left(\frac{2x+4}{3}\right)+2\left(\frac{2x+4}{3}\right)+3 = 100$,

or, $9x = 279$.

Whence, $x = 31$, upper piece; $\frac{2x+4}{3} = 22$, lower piece;

and $2\left(\frac{2x+4}{3}\right)+3 = 47$, middle piece.

66. Let $x =$ No. of miles he can go.

Then will $\frac{x}{7+3} =$ time down, and $\frac{x}{7-3} =$ time back.

Hence, $\frac{x}{10}+\frac{x}{4} = 7$.

Whence, $x = 20$, No. of miles he can row.

67. Let $4x =$ No. of acres of corn.

Then will $3x =$ No. of acres of potatoes.

Hence, $4x-6 : \frac{1}{3}(3x)+15\frac{1}{3} :: 3 : 10$.

From this, $40x-60 = \frac{9x}{2} + \frac{93}{2}$.

Whence, $x = 3$;

$4x = 12$, No. of acres of corn;

and $3x = 9$, No. of acres of potatoes.

68. Let $x =$ No. of \$.50 pieces.

Then will $16-x =$ No. of \$.25 pieces.

Hence, $\$.50x + \$.25(16-x) = \$5.75$.

Whence, $x = 7$, No. of \$.50 pieces;

and $16-x = 9$, No. of \$.25 pieces.

69. Let $x =$ No. of acres in the farm.

Then will $\$75x =$ cost of farm.

Hence, $\$80(x-10) = \$75x + \$40$.

Whence, $x = 168$, No. of acres.

Art. 206.

1. $2x+3y = 23$; (1)

$5x-2y = 10$. (2)

From (1), $x = \frac{23-3y}{2}$;

from (2), $x = \frac{10+2y}{5}$.

$$\frac{23-3y}{2} = \frac{10+2y}{5};$$

$115-15y = 20+4y$.

$19y = 95$;

$y = 5$.

Hence, $x = 4$.

2. $6x-2y = 30$; (1)

$5x-6y = 12$. (2)

From (1), $x = \frac{15+y}{3}$;

from (2), $x = \frac{12+6y}{5}$.

$$\frac{15+y}{3} = \frac{12+6y}{5};$$

$75+5y = 36+18y$.

$13y = 39$;

$y = 3$.

Hence, $x = 6$.

13. $$2x+\frac{y-2}{5} = 21; \quad (1)$$

$$4y+\frac{x-4}{6} = 29. \quad (2)$$

From (1), $x = \dfrac{107-y}{10}$;

from (2), $x = 178-24y$.

$$\frac{107-y}{10} = 178-24y;$$

$$107-y = 1780-240y.$$

$$239y = 1673;$$

$$y = 7.$$

Hence, $x = 10$.

14. $$\frac{3x}{19}+5y = 13; \quad (1)$$

$$2x + \frac{4-7y}{2} = 33. \quad (2)$$

From (1), $x = \dfrac{247-95y}{3}$;

from (2), $x = \dfrac{62+7y}{4}$.

$$\frac{247-95y}{3} = \frac{62+7y}{4};$$

$$988-380y = 186+21y.$$

$$401y = 802;$$

$$y = 2.$$

Hence, $x = 19$.

15. $$\frac{x}{7} + \frac{y}{14} = 10\tfrac{1}{2}; \quad (1)$$

$$2x-y = 7. \quad (2)$$

From (1), $x = \dfrac{147-y}{2}$;

from (2), $x = \dfrac{7+y}{2}$.

$$\frac{147-y}{2} = \frac{7+y}{2};$$

$$147-y = 7+y.$$

$$2y = 140;$$

$$y = 70.$$

Hence, $x = 38\tfrac{1}{2}$.

16. $$\frac{x+y}{3} - \frac{x-y}{2} = 9; \quad (1)$$

$$\frac{x}{2} + \frac{x+y}{9} = 5. \quad (2)$$

From (1), $x = 5y-54$;

from (2), $x = \dfrac{90-2y}{11}$.

$$5y-54 = \frac{90-2y}{11};$$

$$55y-594 = 90-2y.$$

$$57y = 684;$$

$$y = 12.$$

Hence, $x = 6$.

17. $$\frac{3x}{4} - \frac{2y}{3} = 1; \quad (1)$$

$$\frac{7x}{8} + \frac{5y}{6} = 6. \quad (2)$$

From (1), $x = \dfrac{12+8y}{9}$;

from (2), $x = \dfrac{36-5y}{14}$.

$$\frac{12+8y}{9} = \frac{36-5y}{14};$$

$$168+112y = 324-45y.$$

$$157y = 156;$$

$$y = \tfrac{156}{157}.$$

Hence, $x = \tfrac{348}{157}.$

18. $$\frac{x+y}{3}+x = 15; \qquad (1)$$

$$\frac{x-y}{5}+y = 6. \qquad (2)$$

From (1), $x = \dfrac{45-y}{4};$

from (2), $x = 30-4y.$

$$\frac{45-y}{4} = 30-4y;$$

$$45-y = 120-16y.$$

$$15y = 75;$$

$$y = 5.$$

Hence, $x = 10.$

19. $$\frac{7x}{6}+\frac{5y}{3} = 34; \qquad (1)$$

$$\frac{7x}{8}+\frac{3y}{4} = \frac{5y}{8}+12. \qquad (2)$$

From (1), $x = \dfrac{204-10y}{7};$

from (2), $x = \dfrac{96-y}{7}.$

$$\frac{204-10y}{7} = \frac{96-y}{7};$$

$$204-10y = 96-y.$$

$$9y = 108;$$

$$y = 12.$$

Hence, $x = 12.$

20. $$\frac{x+y}{8}-\frac{y-x}{6} = 5; \qquad (1)$$

$$\frac{x+y}{4}-\frac{x-y}{3} = 10. \qquad (2)$$

From (1), $x = \dfrac{120+y}{7};$

from (2), $x = 7y-120.$

$$\frac{120+y}{7} = 7y-120;$$

$$120+y = 49y-840.$$

$$48y = 960;$$

$$y = 20.$$

Hence, $x = 20.$

21. $$\frac{7x}{4}+\frac{5y}{8} = 20; \qquad (1)$$

$$\frac{3x}{5}+\frac{7y}{4} = 2x-7. \qquad (2)$$

From (1), $x = \dfrac{160-5y}{14};$

from (2), $x = \dfrac{140+35y}{28}.$

$$\frac{160-5y}{14} = \frac{140+35y}{28};$$

$$320-10y = 140+35y.$$

$$45y = 180;$$

$$y = 4.$$

Hence, $x = 10.$

22. $$\frac{2x+3y}{5} = 10 - \frac{y}{3}; \qquad (1)$$

$$\frac{4y-3x}{6} = \frac{3x}{4} + 1. \qquad (2)$$

From (1), $x = \dfrac{75-7y}{3}$;

from (2), $x = \dfrac{8y-12}{15}$.

$$\frac{75-7y}{3} = \frac{8y-12}{15};$$

$$375-35y = 8y-12.$$

$$43y = 387;$$

$$y = 9.$$

Hence, $x = 4$.

23. $$\frac{1-3x}{7} - \frac{1-3y}{5} = 2; \qquad (1)$$

$$\frac{3x+y}{11} + y = 9. \qquad (2)$$

From (1), $x = \dfrac{7y-24}{5}$;

from (2), $x = 33-4y$.

$$\frac{7y-24}{5} = 33-4y;$$

$$7y-24 = 165-20y.$$

$$27y = 189;$$

$$y = 7.$$

Hence, $x = 5$.

24. $2(2x+3y) = 3(2x-3y)+10$; (1)

$4x-3y = 4(6y-2x)+3$. (2)

From (1), $x = \dfrac{15y-10}{2}$;

from (2), $x = \dfrac{1+9y}{4}$.

$$\frac{15y-10}{2} = \frac{1+9y}{4};$$

$$30y-20 = 1+9y.$$

$$21y = 21;$$

$$y = 1.$$

Hence, $x = 2\frac{1}{2}$.

25. $$\frac{9}{x} - \frac{4}{y} = 1; \qquad (1)$$

$$\frac{18}{x} + \frac{20}{y} = 16. \qquad (2)$$

From (1), $\dfrac{9}{x} = 1 + \dfrac{4}{y}$;

from (2), $\dfrac{9}{x} = 8 - \dfrac{10}{y}$.

$$1 + \frac{4}{y} = 8 - \frac{10}{y};$$

$$y+4 = 8y-10.$$

$$7y = 14;$$

$$y = 2.$$

Hence, $x = 3$.

26. $$x-4y = 7; \qquad (1)$$

$$\frac{x}{3y} + \frac{11}{10} = \frac{4x-5y}{5y}. \qquad (2)$$

From (1), $x = 7+4y$;

from (2), $x = \frac{9}{2}y$.

$$7+4y = \tfrac{9}{2}y;$$

$$14+8y = 9y.$$

$$y = 14.$$

Hence, $x = 63$.

Art. 208.

1.
$$3x-4y = 2; \qquad (1)$$
$$7x-9y = 7. \qquad (2)$$

From (1). $x = \dfrac{2+4y}{3}$;

and $7x = \dfrac{14+28y}{3}$.

Putting $\dfrac{14+28y}{3}$ for $7x$ in (2),

$$\frac{14+28y}{3} - 9y = 7. \qquad (3)$$

From (3),

$$14+28y-27y = 21.$$

Or, $y = 7$.

Hence, $x = 10$.

2.
$$5x+2y = 29; \qquad (1)$$
$$6x-2y = 26. \qquad (2)$$

From (1), $x = \dfrac{29-2y}{5}$;

and $6x = \dfrac{174-12y}{5}$.

Putting $\dfrac{174-12y}{5}$ for $6x$ in (2),

$$\frac{174-12y}{5} - 2y = 26. \qquad (3)$$

From (3),

$$174-12y-10y = 130.$$

Or, $y = 2$.

Hence, $x = 5$.

3.
$$3x+2y = 0; \qquad (1)$$
$$3x-4y = 18. \qquad (2)$$

From (1), $x = \dfrac{-2y}{3}$;

and $3x = -2y$.

Putting $-2y$ for $3x$ in (2),

$$-2y - 4y = 18.$$

From (3), $-6y = 18$.

Or, $y = -3$.

Hence, $x = 2$.

4.
$$5x-2y = -4; \qquad (1)$$
$$4x+y = 28. \qquad (2)$$

From (1), $x = \dfrac{2y-4}{5}$;

and $4x = \dfrac{8y-16}{5}$.

Putting $\dfrac{8y-16}{5}$ for $4x$ in (2),

$$\frac{8y-16}{5} + y = 28. \qquad (3)$$

From (3),

$$8y-16+5y = 140.$$

Or, $y = 12$.

Hence, $x = 4$.

5. $$2x-3y = 0; \qquad (1)$$
$$8x-y = 22. \qquad (2)$$

From (1), $x = \dfrac{3y}{2}$;

and $8x = 12y$.

Putting $12y$ for $8x$ in (2),
$$12y-y = 22. \qquad (3)$$

From (3), $11y = 22$

Or, $y = 2$.

Hence, $x = 3$.

6. $$6x-5y = 1; \qquad (1)$$
$$14x-8y = 17. \qquad (2)$$

From (1), $x = \dfrac{1+5y}{6}$;

and $14x = \dfrac{14+70y}{6}$.

Putting $\dfrac{14+70y}{6}$ for $14x$ in (2),
$$\frac{14+70y}{6} - 8y = 17. \qquad (3)$$

From (3),
$$14+70y-48y = 102.$$

Or, $y = 4$.

Hence, $x = 3\frac{1}{2}$.

7. $$4x+10y = 5; \qquad (1)$$
$$100x+40y = 41. \qquad (2)$$

From (1), $x = \dfrac{5-10y}{4}$;

and $100x = 125-250y$.

Putting $125-250y$ for $100x$ in (2),
$$125-250y+40y = 41. \qquad (3)$$

From (3), $210y = 84$.

Or, $y = \frac{2}{5}$.

Hence, $x = \frac{1}{4}$.

8. $$\frac{3x}{19} + 5y = 13; \qquad (1)$$
$$2x - \frac{7y-4}{2} = 33. \qquad (2)$$

From (1), $x = \dfrac{247-95y}{3}$;

and $2x = \dfrac{494-190y}{3}$.

Putting $\dfrac{494-190y}{3}$ for $2x$ in (2),
$$\frac{494-190y}{3} - \frac{7y-4}{2} = 33. \qquad (3)$$

From (3),
$$988-380y-21y+12 = 198.$$

Or, $y = 2$.

Hence, $x = 19$.

9. $$\frac{x+y}{3} - \frac{x-y}{2} = 9; \qquad (1)$$
$$\frac{x}{2} + \frac{x+y}{9} = 5. \qquad (2)$$

Or, $$-x+5y = 54; \qquad (1)$$
$$11x+2y = 90. \qquad (2)$$

From (1), $x = 5y-54$;

and $11x = 55y-594$.

Putting $55y-594$ for $11x$ in (2),
$$55y-594+2y = 90. \qquad (3)$$

From (3), $y = 12$.

Hence, $x = 6$.

10. $4x+5y = 40(x-y);$ (1)

$2x+5y = 1\tfrac{1}{2}.$ (2)

From (1), $x = \frac{5}{4}y;$

and $2x = \frac{5}{2}y.$

Putting $\frac{5}{2}y$ for $2x$ in (2),

$\frac{5}{2}y+5y = \frac{3}{2}.$ (3)

From (3), $y = \frac{1}{5}.$

Hence, $x = \frac{1}{4}.$

11. $\frac{x}{3}-\frac{y}{5} = 5;$ (1)

$\frac{x}{5}+y = 31.$ (2)

Or, $5x-3y = 75;$ (1)

$x+5y = 155.$ (2)

From (1), $x = \frac{75+3y}{5}.$

Putting $\frac{75+3y}{5}$ for x in (2),

$\frac{75+3y}{5}+5y = 155.$ (3)

From (3),

$75+3y+25 = 775.$

Or, $y = 25.$

Hence, $x = 30.$

12. $\frac{x}{7}-\frac{7y}{10} = -20;$ (1)

$\frac{x}{4}+3y = 134.$ (2)

Or, $10x-49y = -1400;$ (1)

$x+12y = 536.$ (2)

From (1), $x = \frac{49y-1400}{10}.$

Putting $\frac{49y-1400}{10}$ for x in (2),

$\frac{49y-1400}{10}+12y = 536.$ (3)

From (3),

$49y-1400+120y = 5360.$

Or, $y = 40.$

Hence, $x = 56.$

13. $\frac{5x-6y}{13}+3x = 4y-2;$ (1)

$\frac{5x+6y}{6}-\frac{3x-2y}{4} = 2y-2.$ (2)

Or, $44x-58y = -26;$ (1)

$2x-12y = -48.$ (2)

From (1), $x = \frac{29y-13}{22};$

and $2x = \frac{29y-13}{11}.$

Putting $\frac{29y-13}{11}$ for $2x$ in (2),

$\frac{29y-13}{11}-12y = -48.$ (3)

From (3),

$29y-13-132y = -528.$

Or, $y = 5.$

Hence, $x = 6.$

14. $$2x - \frac{x-3}{2} = 6; \quad (1)$$

$$5x + 4y - \frac{7y+5x}{2} = 10. \quad (2)$$

Or, $$3x = 9; \quad (1)$$

$$5x + y = 20. \quad (2)$$

From (1), $x = 3$;

and $5x = 15$.

Putting 15 for $5x$ in (2),

$$15 + y = 20. \quad (3)$$

From (3), $y = 5$.

15. $$\frac{x}{15} - \frac{y}{20} = 1; \quad (1)$$

$$\frac{y}{10} - \frac{x}{30} = 1. \quad (2)$$

Or, $$4x - 3y = 60; \quad (1)$$

$$3y - x = 30. \quad (2)$$

From (1), $x = \frac{60+3y}{4}$.

Putting $\frac{60+3y}{4}$ for x in (2),

$$3y - \frac{60+3y}{4} = 30. \quad (3)$$

From (3), $y = 20$.

Hence, $x = 30$.

16. $$\frac{x}{2} + \frac{y}{3} = 7; \quad (1)$$

$$\frac{x}{3} + \frac{y}{2} = 8. \quad (2)$$

Or, $$3x + 2y = 42; \quad (1)$$

$$2x + 3y = 48. \quad (2)$$

From (1), $x = \frac{42-2y}{3}$;

and $2x = \frac{84-4y}{3}$.

Putting $\frac{84-4y}{3}$ for $2x$ in (2),

$$\frac{84-4y}{3} + 3y = 48. \quad (3)$$

From (3), $y = 12$.

Hence, $x = 6$.

17. $$\frac{4x+3y}{6} = 8; \quad (1)$$

$$\frac{7y-3x}{2} - y = 11. \quad (2)$$

Or, $$4x + 3y = 48; \quad (1)$$

$$5y - 3x = 22. \quad (2)$$

From (1), $x = \frac{48-3y}{4}$;

and $3x = \frac{144-9y}{4}$.

Putting $\frac{144-9y}{4}$ for $3x$ in (2),

$$5y - \frac{144-9y}{4} = 22. \quad (3)$$

From (3), $y = 8$.

Hence, $x = 4$.

18.

$$\frac{x}{6} + 6y = 150; \quad (1)$$

$$6x + \frac{y}{6} = 220. \quad (2)$$

Or, $$x + 36y = 900; \quad (1)$$

$$36x + y = 1320. \quad (2)$$

From (1), $x = 900 - 36y$;

and $36x = 32400 - 1296y$.

Putting $32400 - 1296y$ for $36x$ in (2),

$$32400 - 1296y + y = 1320. \quad (3)$$

From (3), $y = 24$.

Hence, $x = 36$.

19.

$$\frac{74}{x} - \frac{74}{y} = 35; \quad (1)$$

$$\frac{19}{x} + \frac{37}{y} = 10\tfrac{1}{2}. \quad (2)$$

Or, $$\frac{37}{x} - \frac{37}{y} = \frac{35}{2}; \quad (1)$$

$$\frac{19}{x} + \frac{37}{y} = \frac{21}{2}. \quad (2)$$

From (1), $\dfrac{37}{y} = \dfrac{37}{x} - \dfrac{35}{2}$.

Putting $\dfrac{37}{x} - \dfrac{35}{2}$ for $\dfrac{37}{y}$ in (2),

$$\frac{19}{x} + \frac{37}{x} - \frac{35}{2} = \frac{21}{2}. \quad (3)$$

From (3),

$$38 + 74 - 35x = 21x.$$

Or, $x = 2$.

Hence, $y = 37$.

20.

$$\frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4}; \quad (1)$$

$$\frac{2x+4}{3} - \frac{2x+y}{8} = \frac{8+5}{4}. \quad (2)$$

Or, $$32x - 15y = 74; \quad (1)$$

$$4x - 3y = -2. \quad (2)$$

From (1), $x = \dfrac{74 + 15y}{32}$;

and $4x = \dfrac{74 + 15y}{8}$.

Putting $\dfrac{74 + 15y}{8}$ for $4x$ in (2),

$$\frac{74 + 15y}{8} - 3y = -2. \quad (3)$$

From (3), $y = 10$.

Hence, $x = 7$.

21.

$$\frac{7+x}{5} - \frac{2x-y}{4} = 3y - 5; \quad (1)$$

$$\frac{5y-7}{2} - \frac{3-4x}{6} = 18 - 5x. \quad (2)$$

Or, $$6x + 55y = 128; \quad (1)$$

$$34x + 15y = 132. \quad (2)$$

From (1), $= \dfrac{128 - 55y}{6}$;

and $34x = \dfrac{2176 - 935y}{3}$.

Putting $\dfrac{2176 - 935y}{3}$ for $34x$ in (2),

$$\frac{2176 - 935y}{3} + 15y = 132. \quad (3)$$

From (3), $y = 2$.

Hence, $x = 3$.

22. $\dfrac{x+y}{2}-\dfrac{x-y}{3}=8;$ (1)

$\dfrac{x+y}{3}-\dfrac{y-x}{4}=11.$ (2)

Or, $x+5y=48;$ (1)

$7x+y=132.$ (2)

From (1), $x=48-5y;$

and $7x=336-35y.$

Putting $336-35y$ for $7x$ in (2),

$336-35y+y=132.$ (3)

From (3), $y=6.$

Hence, $x=18.$

23. $\dfrac{x+y}{3}-\dfrac{y-x}{4}=59;$ (1)

$\dfrac{x}{11}=\dfrac{3y}{5}.$ (2)

Or, $7x+y=708;$ (1)

$5x-33y=0.$ (2)

From (1), $x=\dfrac{708-y}{7};$

and $5x=\dfrac{3540-5y}{7}.$

Putting $\dfrac{3540-5y}{7}$ for $5x$ in (2),

$\dfrac{3540-5y}{7}-33y=0.$ (3)

From (3), $y=15.$

Hence, $x=99.$

24. $\dfrac{x}{2}-\dfrac{y}{3}-1=0;$ (1)

$\dfrac{2x-1}{2}-\dfrac{3y-1}{3}=\dfrac{5}{6}.$ (2)

Or, $3x-2y=6;$ (1)

$x-y=1.$ (2)

From (1), $x=\dfrac{6+2y}{3}.$

Putting $\dfrac{6+2y}{3}$ for x in (2),

$\dfrac{6+2y}{3}-y=1.$ (3)

From (3), $y=3.$

Hence, $x=4.$

25. $\dfrac{3x-5y}{3}-\dfrac{2x-8y-9}{12}$

$=\dfrac{y}{2}+\dfrac{7}{12};$ (1)

$3\frac{1}{2}\left(\dfrac{x}{7}+\dfrac{y}{4}+1\frac{1}{3}\right)-3\frac{1}{3}\left(4x-\dfrac{y}{8}-24\right)$

$=0.$ (2)

Or, $5x-9y=-1;$ (1)

$308x-31y=2032.$ (2)

From (1), $x=\dfrac{9y-1}{5};$

and $308x=\dfrac{2772y-308}{5}.$

Putting $\dfrac{2772y-308}{5}$ for $308x$ in (2),

$\dfrac{2772y-308}{5}-31y=2032.$ (3)

From (3), $y=4.$

Hence, $x=7.$

Art. 211.

1.

$$7x-2y = 68; \qquad (1)$$
$$4x-5y = 8. \qquad (2)$$

$$\begin{aligned}
(1)\times 4 &= 28x-\ \ 8y = 272.\\
(2)\times 7 &= \underline{28x-35y = \ \ 56.}\\
\text{Subtracting,}\quad 27y &= 216,\\
y &= 8.\\
\text{Hence,}\quad x &= 12.
\end{aligned}$$

2.

$$3x+2y = 26; \qquad (1)$$
$$\underline{5x-2y = 38.} \qquad (2)$$

$$\begin{aligned}
\text{Adding,}\quad 8x \quad &= 64,\\
x &= 8.\\
\text{Hence,}\quad y &= 1.
\end{aligned}$$

3.

$$4y+x = 102; \qquad (1)$$
$$y+4x = 48. \qquad (2)$$

$$\begin{aligned}
(1)\times 4 &= 16y+4x = 408.\\
(2) &= \underline{\ \ \ \ y+4x = \ \ 48.}\\
\text{Subtracting,}\ 15y \quad &= 360,\\
y &= 24.\\
\text{Hence,}\quad x &= 6.
\end{aligned}$$

4.

$$2x-y = 6; \qquad (1)$$
$$4x+3y = 22. \qquad (2)$$

$$\begin{aligned}
(1)\times 2 &= 4x-2y = 12.\\
(2) &= \underline{4x+3y = 22.}\\
\text{Subtracting,}\quad -5y &= -10,\\
y &= 2.\\
\text{Hence,}\quad x &= 4.
\end{aligned}$$

5.

$$2x+3y = 23; \qquad (1)$$
$$5x-2y = 10. \qquad (2)$$

$$\begin{aligned}
(1)\times 5 &= 10x+15y = 115.\\
(2)\times 2 &= \underline{10x-\ \ 4y = \ \ 20.}\\
\text{Subtracting,}\quad 19y &= \ \ 95,\\
y &= \ \ \ 5.\\
\text{Hence,}\quad x &= \ \ \ 4.
\end{aligned}$$

6.

$$3x+4y = 188; \qquad (1)$$
$$2x+5y = 207. \qquad (2)$$

$$\begin{aligned}
(1)\times 2 &= \ \ 6x+\ \ 8y = 376.\\
(2)\times 3 &= \underline{\ \ 6x+15y = 621.}\\
\text{Subtracting,}\ -\ 7y &= -245,\\
y &= 35.\\
\text{Hence,}\quad x &= 16.
\end{aligned}$$

7.

$$2x+7y = 34; \qquad (1)$$
$$5x+9y = 51. \qquad (2)$$

$$\begin{aligned}
(1)\times 5 &= 10x+35y = 170.\\
(2)\times 2 &= \underline{10x+18y = 102.}\\
\text{Subtracting,}\quad 17y &= \ \ 68;\\
y &= 4.\\
\text{Hence,}\quad x &= 8.
\end{aligned}$$

8.

$$x+4y = 16; \qquad (1)$$
$$4x+y = 34. \qquad (2)$$

$$\begin{aligned}
(1)\times 4 &= 4x+16y = 64.\\
(2) &= \underline{4x+\ \ \ \ y = 34.}\\
\text{Subtracting,}\quad 15y &= 30,\\
y &= 2.\\
\text{Hence,}\quad x &= 8.
\end{aligned}$$

9. $12x+13y = 37;$ (1)

$17x-19y = 15.$ (2)

$(1)\times 17 = 204x+221y = 629.$

$(2)\times 12 = \underline{204x-228y = 180.}$

Subtracting, $449y = 449,$

$y = 1.$

Hence, $x = 2.$

10. $2x-9y = 11;$ (1)

$x-4y = 5.$ (2)

$(1) = 2x-9y = 11.$

$(2)\times 2 = \underline{2x-8y = 10.}$

Subtracting, $-y = 1,$

$y = -1.$

Hence, $x = 1.$

11. $9x-4y = 8;$ (1)

$13x+7y = 101.$ (2)

$(1)\times 7 = 63x-28y = 56.$

$(2)\times 4 = \underline{52x+28y = 404.}$

Adding, $115x = 460,$

$x = 4.$

Hence, $y = 7.$

12. $3y-x = 104;$ (1)

$5x-2y = 78.$ (2)

$(1)\times 5 = 15y-5x = 520.$

$(2) = \underline{-2y+5x = 78.}$

Adding, $13y = 598,$

$y = 46.$

Hence, $x = 34.$

13. $7y-3x = -7;$ (1)

$11x+5y = 87.$ (2)

$(1)\times 11 = 77y-33x = -77.$

$(2)\times 3 = \underline{15y+33x = 261.}$

Adding, $92y = 184,$

$y = 2.$

Hence, $x = 7.$

14. $5x-3y = 9;$ (1)

$5y+2x = 16.$ (2)

$(1)\times 2 = 10x-6y = 18.$

$(2)\times 5 = \underline{10x+25y = 80.}$

Subtracting, $-31y = -62,$

$y = 2.$

Hence, $x = 3.$

15. $2x+3y = 32;$ (1)

$11y-9x = 3.$ (2)

$(1)\times 9 = 18x+27y = 288.$

$(2)\times 2 = \underline{-18x+22y = 6.}$

Adding, $49y = 294,$

$y = 6.$

Hence, $x = 7.$

16. $x - \dfrac{2y-x}{23-x} = 20 - \dfrac{59-2x}{2};$ (1)

$y + \dfrac{y-3}{x-18} = 30 - \dfrac{73-3y}{3}.$ (2)

Or, $17x+4y = 437;$ (1)

$3y-17x = -297.$ (2)

$(1) = 17x+4y = 437.$

$(2) = \underline{-17x+3y = -297.}$

Adding, $7y = 140,$

$y = 20.$

Hence, $x = 21.$

17.
$$\frac{6y+9}{4} + \frac{3y-5}{4x-6} = 1\tfrac{1}{4} + \frac{3y+4}{2}; \quad (1)$$
$$\frac{8y+7}{10} - \frac{3y-6x}{2y-8} = 4 - \frac{9-4y}{5}. \quad (2)$$

Or,
$$8x-6y = 2; \quad (1)$$
$$x-y = -2. \quad (2)$$

$$\begin{array}{ll} (1) = & 8x-6y = 2. \\ (2)\times 8 = & 8x-8y = -16. \\ \textbf{Subtracting,} & 2y = 18, \\ & y = 9. \qquad \text{Hence, } x = 7. \end{array}$$

18.
$$\frac{x}{3y} = 1\tfrac{1}{3}; \quad (1)$$
$$\tfrac{1}{5}(2x+7y)-1 = \tfrac{2}{3}(2x-6y+1). \quad (2)$$

Or,
$$x-4y = 0; \quad (1)$$
$$-14x+81y = 25. \quad (2)$$

$$\begin{array}{ll} (1)\times 14 = & 14x-56y = 0. \\ (2) = & -14x+81y = 25. \\ \textbf{Adding,} & 25y = 25, \\ & y = 1. \qquad \text{Hence, } x = 4. \end{array}$$

19.
$$y-3x = 1; \quad (1)$$
$$2(3x+y) = 3(y-x)+1. \quad (2)$$

Or,
$$-3x+y = 1; \quad (1)$$
$$9x-y = 1. \quad (2)$$

$$\begin{array}{ll} (1)\times 3 = & -9x+3y = 3. \\ (2) = & 9x-y = 1. \\ \textbf{Adding,} & 2y = 4, \\ & y = 2. \\ \textbf{Hence,} & x = \tfrac{1}{3}. \end{array}$$

20.
$$x + \frac{y}{2} = 17; \quad (1)$$
$$\frac{2y-x}{3x+y} = \frac{13}{45}. \quad (2)$$

Or,
$$2x+y = 34; \quad (1)$$
$$11y-12x = 0. \quad (2)$$

$$\begin{array}{ll} (1)\times 6 = & 12x+6y = 204. \\ (2) = & -12x+11y = 0. \\ \textbf{Adding,} & 17y = 204, \\ & y = 12. \\ \textbf{Hence,} & x = 11. \end{array}$$

21.

$$56(1-x-y) = 5; \quad (1)$$
$$51(y-x) = 19(x+y). \quad (2)$$

Or,
$$56x+56y = 51; \quad (1)$$
$$16y-35x = 0. \quad (2)$$

$$\begin{array}{lrl}
(1)\times 2 = & 112x+112y & = 102. \\
(2)\times 7 = & -245x+112y & = 0. \\
\text{Subtracting,} & 357x & = 102,
\end{array}$$

$$x = \tfrac{2}{7}. \quad \text{Hence,} \quad y = \tfrac{5}{8}.$$

22.

$$\frac{1}{x}+\frac{1}{y} = \frac{8}{9}; \quad (1)$$
$$4x-3y = 2xy. \quad (2)$$

$$(2)\div xy = \frac{4}{y}-\frac{3}{x} = 2.$$

$$(1)\times 3 = \frac{3}{y}+\frac{3}{x} = \frac{8}{3}.$$

$$\text{Adding,} \quad \frac{7}{y} = \frac{14}{3},$$

$$y = 1\tfrac{1}{2}.$$

Hence, $x = 4\tfrac{1}{2}$.

23.

$$22x-y = 33; \quad (1)$$
$$\frac{y+x}{5x-y}+\frac{5}{8} = 0. \quad (2)$$

Or,
$$22x-y = 33; \quad (1)$$
$$33x+3y = 0. \quad (2)$$

$$\begin{array}{lrl}
(1)\times 3 = & 66x-3y & = 99. \\
(2) = & 33x+3y & = 0. \\
\text{Adding,} & 99x & = 99,
\end{array}$$

$$x = 1.$$

Hence, $y = -11$.

24.

$$\frac{y+x}{y-x} = 12; \quad (1)$$
$$\frac{3x-y}{2}-\frac{2y-x}{5} = 7. \quad (2)$$

Or,
$$13x-11y = 0; \quad (1)$$
$$17x-9y = 70. \quad (2)$$

$$\begin{array}{lrl}
(1)\times 9 = & 117x-99y & = 0. \\
(2)\times 11 = & 187x-99y & = 770. \\
\text{Subtracting,} & -70x & = -770,
\end{array}$$

$$x = 11. \quad \text{Hence,} \quad y = 13.$$

25. $5(x+1)+3(2y-5) = 2$; (1) $\qquad 4x-10y = 17$. (2)

Or, $5x+6y = 12$. (1)

$$\begin{aligned}
(1)\times 4 = \quad && 20x+24y &= 48. \\
(2)\times 5 = \quad && 20x-50y &= 85. \\
\text{Subtracting,} \quad && 74y &= -37, \\
&& y &= -\tfrac{1}{2}. \qquad \text{Hence,} \quad x = 3.
\end{aligned}$$

26. $\dfrac{1}{2y}-\dfrac{1}{3x}=\dfrac{7}{15}$; (1) $\qquad \dfrac{y-5x+4xy}{4xy}=\dfrac{1}{15}$. (2)

Or, $\dfrac{45}{y}-\dfrac{30}{x}=42$; (1) $\qquad \dfrac{15}{x}-\dfrac{75}{y}=-56$. (2)

$$\begin{aligned}
(1) = \quad && \frac{45}{y}-\frac{30}{x} &= 42. \\
(2)\times 2 = \quad && -\frac{150}{y}+\frac{30}{x} &= -112. \\
\text{Adding,} \quad && -\frac{105}{y} &= -70, \\
&& y &= 1\tfrac{1}{2}. \qquad \text{Hence,} \quad x = -2\tfrac{1}{2}.
\end{aligned}$$

Art. 213.

1. $ax+by=c$; (1) $\qquad px+qy=r$. (2)

$$\begin{aligned}
(1)\times p = \quad && apx+bpy &= cp. \\
(2)\times a = \quad && apx+aqy &= ar. \\
\text{Subtracting,} \quad && (aq-bp)\,y &= ar-cp, \\
&& y &= \frac{ar-cp}{aq-bp}. \\
\text{Hence,} \quad && x &= \frac{cq-br}{aq-bp}.
\end{aligned}$$

2. $ax-by=m$; (1) $\qquad cx+ey=n$. (2)

$$\begin{aligned}
(1)\times c = \quad && acx-bcy &= cm. \\
(2)\times a = \quad && acx+aey &= an. \\
\text{Subtracting,} \quad && (ae+bc)\,y &= an-cm,
\end{aligned}$$

$$y = \frac{an-cm}{ae+bc}.$$

Hence, $$x = \frac{cm+bn}{ae+bc}.$$

3. $x+ay = b$; (1) $ax-by = c$. (2)

$(1)\times a =$ $ax+a^2y = ab$.

$(2) =$ $ax-by = c$.

Subtracting, $(a^2+b)\,y = ab-c$,

$$y = \frac{ab-c}{a^2+b}.$$

Hence, $$x = \frac{ac+b^2}{a^2+b}.$$

4. $ax+2by = c$; (1) $a(3a-x) = b(x+y)$. (2)

Or, $(a+b)\,x+by = 3a^2$. (2)

$(1)\times(a+b) =$ $(a+b)\,ax+(a+b)\,2by = ac+bc$.

$(2)\times a =$ $(a+b)\,ax+ \quad aby = 3a^3$.

Subtracting, $(ab+2b^2)\,y = ac+bc-3a^3$,

$$y = \frac{ac+bc-3a^3}{b\,(a+2b)}.$$

Hence, $$x = \frac{6a^2-c}{a+2b}.$$

5. $ax+by = c$; (1) $bx-ay = d$. (2)

$(1)\times b =$ $abx+b^2y = bc$.

$(2)\times a =$ $abx-a^2y = ad$.

Subtracting, $(a^2+b^2)\,y = bc-ad$,

$$y = \frac{bc-ad}{a^2+b^2}. \quad \text{Hence,} \quad x = \frac{ac+bd}{a^2+b^2}.$$

6. $ax-by = 0$; (1) $x+y = c$. (2)

$(1) =$ $ax-by = 0$.

$(2)\times a =$ $ax+ay = ac$.

Subtracting, $(a+b)\,y = ac$,

$$y = \frac{ac}{a+b}.$$

Hence, $$x = \frac{bc}{a+b}.$$

7. $$\frac{x}{a} + \frac{y}{b} = 2; \quad (1) \qquad bx - ay = 0. \quad (2)$$

Or, $$bx + ay = 2ab. \quad (1)$$

$$\begin{aligned}
(1) = \quad & bx + ay = 2ab. \\
(2) = \quad & bx - ay = 0. \\
\text{Subtracting,} \quad & 2ay = 2ab, \\
& y = b. \qquad \text{Hence, } x = a.
\end{aligned}$$

8. $$x + y = a + b; \quad (1) \qquad bx - ay = 0. \quad (2)$$

$$\begin{aligned}
(1) \times a = \quad & ax + ay = a^2 + ab. \\
(2) = \quad & bx - ay = 0. \\
\text{Adding,} \quad & (a+b)x = a^2 + ab, \\
& x = a. \qquad \text{Hence, } y = b.
\end{aligned}$$

9. $$\frac{x}{a} + \frac{y}{b} = 1; \quad (1) \qquad \frac{x}{b} + \frac{y}{a} = 1. \quad (2)$$

Or, $$bx + ay = ab; \quad (1) \qquad ax + by = ab. \quad (2)$$

$$\begin{aligned}
(1) \times a = \quad & abx + a^2y = a^2b. \\
(2) \times b = \quad & abx + b^2y = ab^2. \\
\text{Subtracting,} \quad & (a^2 - b^2)y = a^2b - ab^2, \\
& y = \frac{ab}{a+b}.
\end{aligned}$$

Hence, $$x = \frac{ab}{a+b}.$$

10. $$(a+c)x - by = bc; \quad (1) \qquad x + y = a + b. \quad (2)$$

$$\begin{aligned}
(1) = \quad & (a+c)x - by = bc. \\
(2) \times b = \quad & bx + by = ab + b^2. \\
\text{Adding,} \quad & (a+b+c)x = ab + b^2 + bc, \\
& x = b. \qquad \text{Hence, } y = a.
\end{aligned}$$

11. $\frac{x}{a}+\frac{y}{b}=c$; (1) $\frac{x}{b}-\frac{y}{a}=0$. (2)

Or, $bx+ay=abc$; (1) $ax-by=0$. (2)

$$\begin{aligned}
(1)\times a &= & abx+a^2y &= a^2bc. \\
(2)\times b &= & abx-b^2y &= 0. \\
\text{Subtracting,} && (a^2+b^2)\,y &= a^2bc, \\
&& y &= \frac{a^2bc}{a^2+b^2}. \\
\text{Hence,} && x &= \frac{ab^2c}{a^2+b^2}.
\end{aligned}$$

12. $\frac{m}{x}+\frac{n}{y}=a$; (1) $\frac{n}{x}+\frac{m}{y}=b$. (2)

$$\begin{aligned}
(1)\times n &= & \frac{mn}{x}+\frac{n^2}{y} &= an. \\
(2)\times m &= & \frac{mn}{x}+\frac{m^2}{y} &= bm. \\
\text{Subtracting,} && \frac{n^2-m^2}{y} &= an-bm, \\
&& y &= \frac{n^2-m^2}{an-bm}. \\
\text{Hence,} && x &= \frac{m^2-n^2}{am-bn}.
\end{aligned}$$

13. $x+y=c$; (1) $ax-by=c(a-b)$. (2)

$$\begin{aligned}
(1)\times a &= & ax+ay &= ac. \\
(2) &= & ax-by &= ac-bc. \\
\text{Subtracting,} && (a+b)\,y &= bc, \\
&& y &= \frac{bc}{a+b}. \qquad \text{Hence,}\quad x = \frac{ac}{a+b}.
\end{aligned}$$

14. $a(x+y)+b(x-y)=1$; (1) $a(x-y)+b(x+y)=1$. (2)

$$\begin{aligned}
\text{Or,} && (a+b)\,x+(a-b)\,y &= 1; && (1)\\
&& (a+b)\,x-(a-b)\,y &= 1. && (2)\\
\text{Adding,} && 2(a+b)\,x &= 2, \\
&& x &= \frac{1}{a+b}. \qquad \text{Hence,}\quad y = 0.
\end{aligned}$$

15. $$\frac{x-a}{b}-\frac{b-y}{a}=0; \quad (1)$$

$$\frac{x+y-b}{a}-\frac{a+y-x}{b}=0. \quad (2)$$

Or, $$ax+by=a^2+b^2; \quad (1)$$

$$(a+b)x-(a-b)y=a^2+b^2. \quad (2)$$

$(1)\times(a+b)=$ $$(a^2+ab)x+(ab+b^2)y=a^3+ab^2+a^2b+b^3.$$

$(2)\times a=$ $$(a^2+ab)x+(ab-a^2)y=a^3+ab^2.$$

Subtracting, $$(a^2+b^2)y=a^2b+b^3,$$

$$y=b.$$

Hence, $$x=a.$$

16. $$(a+b)x-(a-b)y=4ab; \quad (1)$$

$$(a-b)x+(a+b)y=2(a^2-b^2). \quad (2)$$

$(1)\times(a-b)=$ $$(a^2-b^2)x-(a-b)^2y=4ab(a-b).$$

$(2)\times(a+b)=$ $$(a^2-b^2)x+(a+b)^2y=2(a+b)^2(a-b).$$

Subtracting, $$2(a^2+b^2)y=2(a^2+b^2)(a-b),$$

$$y=a-b.$$

Hence, $$x=a+b.$$

17. $$\frac{x}{a+b}+\frac{y}{a-b}=2a; \quad (1) \qquad \frac{x-y}{2ab}=\frac{x+y}{a^2+b^2}. \quad (2)$$

Or, $$(a-b)x+(a+b)y=2a(a^2-b^2); \quad (1)$$

$$(a-b)^2x-(a-b)^2y=0. \quad (2)$$

$(1)\times(a-b)=$ $$(a-b)^2x+(a^2-b^2)y=2a(a-b)^2(a+b).$$

$(2)=$ $$(a-b)^2x-(a-b)^2y=0.$$

Subtracting, $$2a(a+b)y=2a(a-b)^2(a+b),$$

$$y=(a-b)^2.$$

Hence, $$x=(a+b)^2.$$

18. $$(a+h)x+(b-h)y=c; \quad (1)$$

$$(b+k)x+(a-k)y=c. \quad (2)$$

$(1)\times(b+k)=$ $$(a+h)(b+k)x+(b^2-bh+bk-kh)y=bc+kc.$$

$(2)\times(a+h)=$ $$(a+h)(b+k)x+(a^2-ak+ah-kh)y=ac+hc.$$

Subtracting, $$(a+b)(-a+b-h+k)y=c(-a+b-h+k),$$

$$y=\frac{c}{a+b}. \qquad \text{Hence,} \quad x=\frac{c}{a+b}.$$

19. $\dfrac{x}{b+c}+\dfrac{y}{a+c}=1$; (1) $\dfrac{ax+cx}{(a+b)c}=1$. (2)

Or, $(a+c)x+(b+c)y = ab+ac+bc+c^2$; (1)

$(a+c)x = ac+bc$. (2)

Subtracting, $(b+c)y = ab+c^2$,

$$y=\frac{ab+c^2}{b+c}.$$

Hence, $$x=\frac{(a+b)c}{a+c}.$$

20. $\dfrac{x}{a+b}+\dfrac{y}{a-b}=2a$; (1) $\dfrac{x-y}{4ab}=1$. (2)

Or, $(a-b)x+(a+b)y = 2a(a^2-b^2)$; (1)

$x-y = 4ab$. (2)

(1) = $(a-b)x+(a+b)y = 2a(a^2-b^2)$,

(2) × $(a-b)$ = $(a-b)x-(a-b)y = 4ab(a-b)$.

Subtracting, $2ay = 2a(a-b)^2$,

$$y=(a-b)^2.$$

Hence, $$x=(a+b)^2.$$

Art. 214.

1. $3x+5y=55$; (1) $5x-y=45$. (2)

(2) × 5 = $25x-5y = 225$.

(1) = $3x+5y = 55$.

Adding, $28x = 280$,

$x=10$. Hence, $y=5$.

2. $5x-11y=19$; (1) $7x+y=43$. (2)

(2) × 11 = $77x+11y = 473$.

(1) = $5x-11y = 19$.

Adding, $82x = 492$,

$x=6$. Hence, $y=1$.

3. $3x+2y=118$; (1) $x+5y=191$. (2)

$$\begin{array}{ll} (2)\times 3 = & 3x+15y=573. \\ (1) = & \underline{3x+\ 2y=118.} \\ \text{Subtracting,} & 13y=455, \\ & y=35. \quad \text{Hence, } x=16. \end{array}$$

4. $2x-y=1$; (1) $x+2y=8$. (2)

$$\begin{array}{ll} (2)\times 2 = & 2x+4y=16. \\ (1) = & \underline{2x-\ y=\ 1.} \\ \text{Subtracting,} & 5y=15, \\ & y=3. \quad \text{Hence, } x=2. \end{array}$$

5. $3x+y=5$; (1) $x+y=3$. (2)

$$\begin{array}{ll} (1) = & 3x+y=5. \\ (2) = & \underline{x+y=3.} \\ \text{Subtracting,} & 2x\quad =2, \\ & x=1. \quad \text{Hence, } y=2. \end{array}$$

6. $x+4y=11$; (1) $x-y=1$. (2)

From (1), $x=11-4y$.

Putting $11-4y$ for x in (2),

$$11-4y-y=1. \qquad (3)$$

From (3), $y=2$. Hence, $x=3$.

7. $x-y=-3$; (1) $3x-y=1$. (2)

$$\begin{array}{ll} (1) = & x-y=-3. \\ (2) = & \underline{3x-y=1.} \\ \text{Subtracting,} & 2x=4, \\ & x=2. \quad \text{Hence, } y=5. \end{array}$$

8. $5x-2y=-1$; (1) $x+y=4$. (2)

$$\begin{array}{ll} (2)\times 5 = & 5x+5y=20. \\ (1) = & \underline{5x-2y=-1.} \\ \text{Subtracting,} & 7y=21, \\ & y=3. \quad \text{Hence, } x=1. \end{array}$$

9. $7x+2\tfrac{1}{2}y = 411\tfrac{1}{2}$; (1) $\qquad 39x-14y = -935\tfrac{9}{10}$. (2)

Or, $14x+5y = 823$; (1) $\qquad 390x-140y = -9359$. (2)

$$\begin{array}{lrl}
(1)\times 28 = & 392x+140y &= 23044. \\
(2) = & 390x-140y &= -9359. \\
\hline
\text{Adding,} & 782x &= 13685, \\
 & x &= 17\tfrac{1}{2}. \\
\text{Hence,} & y &= 115\tfrac{3}{5}.
\end{array}$$

10. $5x-8\tfrac{1}{2} = 7y+21$; (1) $\qquad 2x = 5y-6\tfrac{3}{7}$. (2)

Or, $10x-14y = 59$; (1) $\qquad 14x-35y = -45$. (2)

From (1), $$x = \frac{59+14y}{10},$$

and $$14x = \frac{413+98y}{5}.$$

Putting $\dfrac{413+98y}{5}$ for $14x$ in (2),

$$\frac{413+98y}{5} - 35y = -45. \tag{3}$$

From (3), $y = 8\tfrac{2}{7}$. Hence, $x = 17\tfrac{1}{2}$.

11. $\dfrac{x}{4}-\dfrac{y}{2} = 1$; (1) $\qquad \dfrac{x}{12}+\dfrac{y}{6} = 1$. (2)

Or, $x-2y = 4$; (1) $\qquad x+2y = 12$. (2)

$$\begin{array}{lrl}
(1) = & x-2y &= 4. \\
(2) = & x+2y &= 12. \\
\hline
\text{Adding,} & 2x &= 16, \\
 & x &= 8.
\end{array}$$

Hence, $y = 2$.

12. $32x+81y = 43$; (1) $\qquad 28x-39y = 1$. (2)

$$\begin{array}{lrl}
(1)\times 7 = & 224x+567y &= 301. \\
(2)\times 8 = & 224x-312y &= 8. \\
\hline
\text{Subtracting,} & 879y &= 293, \\
 & y &= \tfrac{1}{3}.
\end{array}$$

Hence, $x = \tfrac{1}{2}$.

13. $96x+75y = 102$; (1) $\qquad 92x+80y = 101.$ (2)

$$\begin{array}{lrl} (1)\times 16 = & 1536x+1200y &= 1632. \\ (2)\times 15 = & \underline{1380x+1200y} &\underline{= 1515.} \\ \text{Subtracting,} & 156x &= 117, \\ & x &= \tfrac{3}{4}. \qquad \text{Hence, } y = \tfrac{2}{5}. \end{array}$$

14. $2x+y = \frac{1}{2}$; (1) $\qquad 18x+22y = -28.$ (2)

$$\begin{array}{lrl} (1)\times 11 = & 44x+22y &= 11. \\ (2) = & \underline{18x+22y} &\underline{= -28.} \\ \text{Subtracting,} & 26x &= 39, \\ & x &= 1\tfrac{1}{2}. \qquad \text{Hence, } y = -2\tfrac{1}{2}. \end{array}$$

15. $11x+9y = 40$; (1) $\qquad \dfrac{3x+y}{8} - \dfrac{2x-y}{31} = 2.$ (2)

Or, $11x+9y = 40$; (1) $\qquad 77x+39y = 496.$ (2)

$$\begin{array}{lrl} (1)\times 7 = & 77x+63y &= 280. \\ (2) = & \underline{77x+39y} &\underline{= 496.} \\ \text{Subtracting,} & -24y &= 216, \\ & y &= -9. \qquad \text{Hence, } x = 11. \end{array}$$

16. $5\frac{3}{4}y-11x = 4y+117\frac{1}{8}$; (1) $\qquad 8x+175 = 2y.$ (2)

Or, $14y-88x = 937$; (1) $\qquad 8x-2y = -175.$ (2)

$$\begin{array}{lrl} (2)\times 11 = & 88x-22y &= -1925. \\ (1) = & \underline{-88x+14y} &\underline{= \quad 937.} \\ \text{Adding,} & -8y &= -988, \\ & y &= 123\tfrac{1}{2}. \qquad \text{Hence, } x = 9. \end{array}$$

17. $13x+7y = 341+7\frac{1}{2}y+43\frac{1}{2}x$; (1) $\qquad 2x+\frac{1}{2}y = 1.$ (2)

Or, $-61x-y = 682$; (1) $\qquad 4x+y = 2.$ (2)

$$\begin{array}{lrl} (1) = & -61x-y &= 682. \\ (2) = & \underline{4x+y} &\underline{= \quad 2.} \\ \text{Adding,} & -57x &= 684, \\ & x &= -12. \qquad \text{Hence, } y = 50. \end{array}$$

18.
$$a(x-a)+b(y-b) = 0; \qquad (1)$$
$$a(x-y-a)+b(x+y-b) = 0. \qquad (2)$$

Or,
$$ax+by = a^2+b^2; \qquad (1)$$
$$(a+b)x-(a-b)y = a^2+b^2. \qquad (2)$$

$$(1)\times(a+b) = \quad (a+b)ax+(ab+b^2)y = a^3+a^2b+ab^2+b^3.$$
$$(2)\times a = \quad (a+b)ax-(a^2-ab)y = a^3+ab^2.$$
Subtracting,
$$(a^2+b^2)y = a^2b+b^3,$$
$$y = b. \qquad \text{Hence,} \quad x = a.$$

19.
$$m(x+y)+n(x-y) = 1; \qquad (1)$$
$$m(x-y)+n(x+y) = 1. \qquad (2)$$

Or,
$$mx+my+nx-ny = 1; \qquad (1)$$
$$mx-my+nx+ny = 1. \qquad (2)$$
Adding,
$$(2m+2n)x = 2,$$
$$x = \frac{1}{m+n}.$$

Subtracting (2) from (1), $(2m-2n)y = 0$, $\qquad y = 0.$

20.
$$bcx = cy-2b; \qquad (1)$$
$$b^2y + \frac{a(c^3-b^3)}{bc} = \frac{2b^3}{c} + c^3x. \qquad (2)$$

Or,
$$bcx-cy = -2b; \qquad (1)$$
$$b^3cy-bc^4x = 2b^4+ab^3-ac^3. \qquad (2)$$

$$(1)\times b^3 = \quad b^4cx-b^3cy = -2b^4.$$
$$(2) = \quad -bc^4x+b^3cy = 2b^4+ab^3-ac^3.$$
Adding,
$$(b^4c-bc^4)x = ab^3-ac^3,$$
$$x = \frac{a}{bc}. \qquad \text{Hence,} \quad y = \frac{a+2b}{c}.$$

Art. 216.

1.

$$\begin{array}{lrr}
 & 2x+3y+4z = 20; & (1)\\
 & 4x-3y-2z = -8; & (2)\\
 & 3x+4y+5z = 26. & (3)\\[6pt]
(1)\times 2 = & 4x+6y+8z = 40. & \\
(2 = & 4x-3y-2z = -8. & \\
\text{Subtracting,} & 9y+10z = 48. & (4)\\[6pt]
(2)\times 3 = & 12x-9y-6z = -24. & \\
(3)\times 4 = & 12x+16y+20z = 104. & \\
\text{Subtracting,} & -25y-26z = -128. & (5)\\[6pt]
(4)\times 13 = & 117y+130z = 624. & \\
(5)\times 5 = & -125y-130z = -640. & \\
\text{Adding,} & -8y = -16, & \\
 & y = 2. & \\
\text{Hence,} & x = 1, \text{ and } z = 3. &
\end{array}$$

2.

$$\begin{array}{lrr}
 & 2x-3y+z = 1; & (1)\\
 & 3x-5y+4z = 3; & (2)\\
 & 4x+2y-3z = 13. & (3)\\[6pt]
(1)\times 3 = & 6x-9y+3z = 3. & \\
(2)\times 2 = & 6x-10y+8z = 6. & \\
\text{Subtracting,} & y-5z = -3. & (4)\\[6pt]
(1)\times 2 = & 4x-6y+2z = 2. & \\
(3) = & 4x+2y-3z = 13. & \\
\text{Subtracting,} & -8y+5z = -11. & (5)\\[6pt]
(4) = & y-5z = -3. & \\
(5) = & -8y+5z = -11. & \\
\text{Adding,} & -7y = -14, & \\
 & y = 2. & \\
\text{Hence,} & x = 3, \text{ and } z = 1. &
\end{array}$$

3.

$$\begin{aligned} x+3y+2z &= 11; && (1)\\ 2x+y+3z &= 14; && (2)\\ 3x+2y+z &= 11. && (3)\end{aligned}$$

$$\begin{array}{lrcll}
(1)\times 2 = & 2x+6y+4z &=& 22. & \\
(2) = & 2x+y+3z &=& 14. & \\
\text{Subtracting,} & 5y+z &=& 8. & (4)\\
(1)\times 3 = & 3x+9y+6z &=& 33. & \\
(3) = & 3x+2y-z &=& 11. & \\
\text{Subtracting,} & 7y+5z &=& 22. & (5)\\
(4)\times 5 = & 25y+5z &=& 40. & \\
(5) = & 7y+5z &=& 22. & \\
\text{Subtracting,} & 18y &=& 18, & \\
 & y &=& 1. & \\
\text{Hence,} & x &=& 2, \quad \text{and} \quad z = 3. &
\end{array}$$

4.

$$\begin{aligned} 5x-6y+4z &= 15; && (1)\\ 7x+4y-3z &= 19; && (2)\\ 2x+y+6z &= 46. && (3)\end{aligned}$$

$$\begin{array}{lrcll}
(1)\times 3 = & 15x-18y+12z &=& 45. & \\
(2)\times 4 = & 28x+16y-12z &=& 76. & \\
\text{Adding,} & 43x-2y &=& 121. & (4)\\
(2)\times 2 = & 14x+8y-6z &=& 38. & \\
(3) = & 2x+y+6z &=& 46. & \\
\text{Adding,} & 16x+9y &=& 84. & (5)\\
(4)\times 9 = & 387x-18y &=& 1089. & \\
(5)\times 2 = & 32x+18y &=& 168. & \\
\text{Adding,} & 419x &=& 1257, & \\
 & x &=& 3. & \\
\text{Hence,} & y &=& 4, \quad \text{and} \quad z = 6. &
\end{array}$$

5.

$$\begin{aligned} 4x-5y+z &= 6; && (1)\\ 7x-11y+2z &= 9; && (2)\\ x+y+3z &= 12. && (3)\end{aligned}$$

(1) × 2 = $8x-10y+2z = 12.$
(2) = $7x-11y+2z = 9.$
Subtracting, $x+y = 3.$ (4)

(2) × 3 = $21x-33y+6z = 27.$
(3) × 2 = $2x+2y+6z = 24.$
Subtracting, $19x-35y = 3.$ (5)

(4) × 35 = $35x+35y = 105.$
(5) = $19x-35y = 3.$
Adding, $54x = 108,$
$x = 2.$
Hence, $y = 1,$ and $z = 3.$

6. $3x-y+z = 17;$ (1)
$5x+3y-2z = 10;$ (2)
$7x+4y-5z = 3.$ (3)

(1) × 2 = $6x-2y+2z = 34.$
(2) = $5x+3y-2z = 10.$
Adding, $11x+y = 44.$ (4)

(1) × 5 = $15x-5y+5z = 85.$
(3) = $7x+4y-5z = 3.$
Adding, $22x-y = 88.$ (5)

(4) = $11x+y = 44.$
(5) = $22x-y = 88.$
Adding, $33x = 132,$
$x = 4.$
Hence, $y = 0,$ and $z = 5.$

7. $x+y+z = 5;$ (1)
$3x-5y+7z = 75;$ (2)
$9x+11y = -10.$ (3)

(1) × 7 = $7x+7y+7z = 35.$
(2) = $3x-5y+7z = 75.$
Subtracting, $4x+12y = -40.$
Or, $x+3y = -10.$ (4)

$$\begin{array}{ll} (4)\times 9 = & 9x+27y = -90. \\ (3) = & 9x+11y = -10. \\ \text{Subtracting,} & 16y = -80, \\ & y = -5. \\ \text{Hence,} & x = 5, \text{ and } z = 5. \end{array}$$

8.

$$\frac{x}{2}+\frac{y}{4}+\frac{z}{6} = 3; \quad (1)$$

$$\frac{x}{4}+y-\frac{z}{12} = 4; \quad (2)$$

$$\frac{x}{5}+y+\frac{z}{10} = 5. \quad (3)$$

$$\begin{array}{llr} \text{Or,} & 6x+3y+2z = 36; & (1) \\ & 3x+12y-z = 48; & (2) \\ & 2x+10y+z = 50. & (3) \end{array}$$

$$\begin{array}{llr} (2)\times 2 = & 6x+24y-2z = 96. & \\ (1) = & 6x+3y+2z = 36. & \\ \text{Adding,} & 12x+27y = 132. & \\ \text{Or,} & 4x+9y = 44. & (4) \\ (2) = & 3x+12y-z = 48. & \\ (3) = & 2x+10y+z = 50. & \\ \text{Adding,} & 5x+22y = 98. & (5) \\ (4)\times 5 = & 20x+45y = 220. & \\ (5)\times 4 = & 20x+88y = 392. & \\ \text{Subtracting,} & 43y = 172, & \\ & y = 4. & \\ \text{Hence,} & x = 2, \text{ and } z = 6. & \end{array}$$

9. $\frac{4x}{5}-\frac{y-3}{2} = 2$; (1) $\frac{x+y+z}{7} = 3$; (2) $\frac{x+y}{4}+\frac{z}{9} = 4$. (3)

$$\begin{array}{llr} \text{Or,} & 8x-5y = 5; & (1) \\ & x+y+z = 21; & (2) \\ & 9x+9y+4z = 144. & (3) \end{array}$$

$(2)\times 9 =$	$9x+9y+9z = 189.$
$(3) =$	$\underline{9x+9y+4z = 144.}$
Subtracting,	$5z = 45,$
	$z = 9.$
Hence,	$x = 5,\quad \text{and}\quad y = 7.$

10. $x+y+z = 29\tfrac{1}{4}$; (1) $\quad x+y-z = 18\tfrac{1}{4}$; (2) $\quad x-y+z = 13\tfrac{3}{4}$. (3)

$(2) =$	$x+y-z = 18\tfrac{1}{4}.$
$(3) =$	$\underline{x-y+z = 13\tfrac{3}{4}.}$
Adding,	$2x = 32,$
	$x = 16.$
Hence,	$y = 7\tfrac{3}{4},\quad \text{and}\quad z = 5\tfrac{1}{2}.$

11. $3x+5y = 161$; (1) $\quad 7x+2z = 209$; (2) $\quad 2y+z = 89$. (3)

$(1)\times 7 =$	$21x+35y = 1127.$	
$(2)\times 3 =$	$\underline{21x+6z = 627.}$	
Subtracting,	$35y-6z = 500.$	(4)
$(3)\times 6 =$	$12y+6z = 534.$	
$(4) =$	$\underline{35y-6z = 500.}$	
Adding,	$47y = 1034,$	
	$y = 22.$	
Hence,	$x = 17,\quad \text{and}\quad z = 45.$	

12. $y+\tfrac{1}{2}x = 41$; (1) $\quad x+\tfrac{1}{4}z = 20\tfrac{1}{2}$; (2) $\quad y+\tfrac{1}{5}z = 34$. (3)

$(1)\times 2 =$	$2y+x = 82.$	
$(2) =$	$\underline{x+\tfrac{1}{4}z = 20\tfrac{1}{2}.}$	
Subtracting,	$2y-\tfrac{1}{4}z = 61\tfrac{1}{2}.$	(4)
$(3)\times 2 =$	$2y+\tfrac{2}{5}z = 68.$	
$(4) =$	$\underline{2y-\tfrac{1}{4}z = 61\tfrac{1}{2}.}$	
Subtracting,	$\tfrac{13}{20}z = 6\tfrac{1}{2},$	
	$z = 10.$	
Hence,	$x = 18,\quad \text{and}\quad y = 32.$	

13. $53-\frac{1}{2}x-\frac{1}{2}z = y-109$; (1) $\quad \frac{1}{4}x+\frac{1}{8}y = 26$; (2) $\quad 5y = 4z$. (3)

Or,
$$\begin{array}{rl}
x+2y+z = 324; & (1)\\
2x+y = 208; & (2)\\
5y-4z = 0. & (3)
\end{array}$$

$$\begin{array}{ll}
(1)\times 2 = & 2x+4y+2z = 648.\\
(2) = & 2x+y = 208.\\
\text{Subtracting,} & 3y+2z = 440. \quad (4)
\end{array}$$

$$\begin{array}{ll}
(4)\times 2 = & 6y+4z = 880.\\
(3) = & 5y-4z = 0.\\
\text{Adding,} & 11y = 880,\\
 & y = 80.
\end{array}$$

Hence, $x = 64$, and $z = 100$.

14.
$$\begin{array}{rl}
2x-\frac{3}{4}y = 93-\frac{1}{2}x-\frac{1}{4}y; & (1)\\
7x-5z = x+y-86; & (2)\\
\frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z = 58. & (3)
\end{array}$$

Or,
$$\begin{array}{rl}
5x-y = 186; & (1)\\
6x-y-5z = -86; & (2)\\
6x+4y+3z = 696. & (3)
\end{array}$$

$$\begin{array}{ll}
(1) = & 5x-y = 186.\\
(2) = & 6x-y-5z = -86.\\
\text{Subtracting,} & -x+5z = 272. \quad (4)
\end{array}$$

$$\begin{array}{ll}
(1)\times 4 = & 20x-4y = 744.\\
(3) = & 6x+4y+3z = 696.\\
\text{Adding,} & 26x+3z = 1440. \quad (5)
\end{array}$$

$$\begin{array}{ll}
(4)\times 3 = & -3x+15z = 816.\\
(5)\times 5 = & 130x+15z = 7200.\\
\text{Subtracting,} & -133x = -6384,\\
 & x = 48.
\end{array}$$

Hence, $y = 54$, and $z = 64$.

15. $\frac{1}{x}+\frac{1}{y}=a;$ (1) $\qquad \frac{1}{x}+\frac{1}{z}=b;$ (2) $\qquad \frac{1}{y}+\frac{1}{z}=c.$ (3)

$$\begin{aligned}
&(1) = && \frac{1}{x}+\frac{1}{y} = a.\\
&(2) = && \frac{1}{x}+\frac{1}{z} = b.\\
&\text{Subtracting,} && \frac{1}{y}-\frac{1}{z} = a-b. \qquad (4)\\
&(3) = && \frac{1}{y}+\frac{1}{z} = c.\\
&(4) = && \frac{1}{y}-\frac{1}{z} = a-b.\\
&\text{Adding,} && \frac{2}{y} = a+c-b,\\
& && y = \frac{2}{a+c-b}.
\end{aligned}$$

Hence, $x=\dfrac{2}{a+b-c}$, and $z=\dfrac{2}{b+c-a}$.

16.

$$\begin{aligned}
x-9y+3z-10u &= 21; && (1)\\
2x+7y-z-u &= 683; && (2)\\
3x+y+5z+2u &= 195; && (3)\\
4x-6y-2z-9u &= 516. && (4)
\end{aligned}$$

$$\begin{aligned}
&(1)\times 2 = && 2x-18y+6z-20u = 42.\\
&(2) = && 2x+7y-z-u = 683.\\
&\text{Subtracting,} && -25y+7z-19u = -641. && (5)\\
&(1)\times 3 = && 3x-27y+9z-30u = 63.\\
&(3) = && 3x+y+5z+2u = 195.\\
&\text{Subtracting,} && -28y+4z-32u = -132. && (6)\\
&(1)\times 4 = && 4x-36y+12z-40u = 84.\\
&(4) = && 4x-6y-2z-9u = 516.\\
&\text{Subtracting,} && -30y+14z-31u = -432. && (7)\\
&(5)\times 2 = && -50y+14z-38u = -1282.\\
&(7) = && -30y+14z-31u = -432.\\
&\text{Subtracting,} && -20y-7u = -850. && (8)
\end{aligned}$$

$$
\begin{array}{lrl}
(6)\times 7 = & -196y+28z-224u = & -924. \\
(7)\times 2 = & -\ 60y+28z-\ 62u = & -864. \\
\text{Subtracting,} & -136y\qquad -162u = & -60. \\
\text{Or,} & -68y\qquad -\ 81u = & -30. \qquad (9) \\
(8)\times 17 = & -340y-119u = & -14450. \\
(9)\times 5 = & -340y-405u = & -\ 150. \\
\text{Subtracting,} & 286u = & -14300, \\
 & u = & -50. \\
\text{Hence,} & z = & -13, \\
 & y = & 60, \quad \text{and} \quad x = 100.
\end{array}
$$

17.

$$
\begin{array}{rl}
x+\ y+\ z+\ u = & 1; \qquad (1) \\
16x+\ 8y+\ 4z+2u = & 9; \qquad (2) \\
81x+27y+\ 9z+3u = & 36; \qquad (3) \\
256x+64y+16z+4u = & 100. \qquad (4)
\end{array}
$$

$$
\begin{array}{lrl}
(1)\times 16 = & 16x+16y+16z+16u = & 16. \\
(2) = & 16x+\ 8y+\ 4z+\ 2u = & 9, \\
\text{Subtracting,} & 8y+12z+14u = & 7. \qquad (5) \\
(1)\times 81 = & 81x+81y+81z+81u = & 81. \\
(3) = & 81x+27y+\ 9z+\ 3u = & 36. \\
\text{Subtracting,} & 54y+72z+78u = & 45. \qquad (6) \\
(1)\times 256 = & 256x+256y+256z+256u = & 256. \\
(4) = & 256x+\ 64y+\ 16z+\ \ 4u = & 100. \\
\text{Subtracting,} & 192y+240z+252u = & 156. \\
\text{Or,} & 16y+\ 20z+\ 21u = & 13. \qquad (7) \\
(5)\times 2 = & 16y+24z+28u = & 14. \\
(7) = & 16y+20z+21u = & 13. \\
\text{Subtracting,} & 4z+\ 7u = & 1. \qquad (8) \\
(7)\times 27 = & 432y+540z+567u = & 351. \\
(6)\times 8 = & 432y+576z+624u = & 360. \\
\text{Subtracting,} & -\ 36z-\ 57u = & -9. \qquad (9)
\end{array}
$$

(8)×9 =	$36z+63u = 9.$
(9) =	$-36z-57u = -9.$
Adding,	$6u = 0,$
	$u = 0.$
Hence,	$z = \frac{1}{4},$
	$y = \frac{1}{2}$, and $x = \frac{1}{4}.$

18. $\frac{x}{3}+\frac{y}{5}+\frac{2z}{7} = 58;$ (1) $\frac{5x}{4}+\frac{y}{6}+\frac{z}{3} = 76;$ (2)

$\frac{x}{2}+\frac{3z}{8}+\frac{u}{5} = 79;$ (3) $y+z+u = 248.$ (4)

Or,	$35x+21y+30z = 6090;$	(1)
	$15x+2y+4z = 912;$	(2)
	$20x+15z+8u = 3160;$	(3)
	$y+z+u = 248.$	(4)
(1)×2 =	$70x+42y+60z = 12180.$	
(2)×21 =	$315x+42y+84z = 19152.$	
Subtracting,	$-245x-24z = -6972.$	(5)
(4)×8 =	$8y+8z+8u = 1984.$	
(3) =	$20x+15z+8u = 3160.$	
Subtracting,	$20x-8y+7z = 1176.$	(6)
(2)×4 =	$60x+8y+16z = 3648.$	
(6) =	$20x-8y+7z = 1176.$	
Adding,	$80x+23z = 4824.$	(7)
(5)×16 =	$3920x+384z = 111552.$	
(7)×49 =	$3920x+1127z = 236376.$	
Subtracting,	$743z = 124824,$	
	$z = 168.$	
Hence,	$u = 50,$	
	$y = 30$, and $x = 12.$	

19.

$$\begin{aligned} 3x-4y+3z-4u &= -8; &&(1)\\ 2x-3y+5z-6u &= -4; &&(2)\\ x+2y-3z+u &= 2; &&(3)\\ 4x-y-2z-u &= -1. &&(4) \end{aligned}$$

(3) × 3 = $3x+6y-9z+3u = 6.$

(1) = $3x-4y+3z-4u = -8.$

Subtracting, $10y-12z+7u = 14.$ (5)

(3) × 2 = $2x+4y-6z+2u = 4.$

(2) = $2x-3y+5z-6u = -4.$

Subtracting, $7y-11z+8u = 8.$ (6)

(3) × 4 = $4x+8y-12z+4u = 8.$

(4) = $4x-y-2z-u = -1.$

Subtracting, $9y-10z+5u = 9.$ (7)

(5) × 7 = $70y-84z+49u = 98.$

(6) × 10 = $70y-110z+80u = 80.$

Subtracting, $26z-31u = 18.$ (8)

(5) × 9 = $90y-108z+63u = 126.$

(7) × 10 = $90y-100z+50u = 90.$

Subtracting, $-8z+13u = 36.$ (9)

(8) × 4 = $104z-124u = 72.$

(9) × 13 = $-104z+169u = 468.$

Adding, $45u = 540,$

$u = 12.$

Hence, $z = 15,$

$y = 11,$ and $x = 13.$

20.

$$\begin{aligned} 5x+4y-3z-2u &= 1; &&(1)\\ x+y+3z-4u &= -2; &&(2)\\ 3x-y+2z-3u &= -5; &&(3)\\ 5x-2y+z-2u &= -1. &&(4) \end{aligned}$$

(1) = $5x+4y-3z-2u = 1.$

(2) = $x+y+3z-4u = -2.$

Adding, $6x+5y-6u = -1.$ (5)

(4) × 2 = $10x-4y+2z-4u=-2.$

(3) = $3x-y+2z-3u=-5.$

Subtracting, $7x-3y-u=3.$ (6)

(4) × 3 = $15x-6y+3z-6u=-3.$

(1) = $5x+4y-3z-2u=1.$

Adding, $20x-2y-8u=-2.$ (7)

(6) × 6 = $42x-18y-6u=18.$

(5) = $6x+5y-6u=-1.$

Subtracting, $36x-23y=19.$ (8)

(6) × 8 = $56x-24y-8u=24.$

(7) = $20x-2y-8u=-2.$

Subtracting, $36x-22y=26.$ (9)

(8) = $36x-23y=19.$

(9) = $36x-22y=26.$

Subtracting, $-y=-7,$

$y=7.$

Hence, $x=5,$

$z=10,$ and $u=11.$

21. $x+y+z-4u=-9;$ (1)

$5x-3y+5z-7u=-16;$ (2)

$3x-4y+4z-2u=-23;$ (3)

$4x-3y-3z-u=-11.$ (4)

(1) × 5 = $5x+5y+5z-20u=-45.$

(2) = $5x-3y+5z-7u=-16.$

Subtracting, $8y-13u=-29.$ (5)

(1) × 4 = $4x+4y-4z-16u=-36.$

(3) = $3x-4y+4z-2u=-23.$

Subtracting, $x+8y-14u=-13.$ (6)

(1) × 3 = $3x+3y+3z-12u=-27.$

(4) = $4x-3y+3z-u=-11.$

Adding, $7x-13u=-38.$ (7)

$$\begin{array}{lrr}
(5) = & 8y - 13u = -29. & \\
(6) = & x + 8y - 14u = -13. & \\
\text{Subtracting,} & -x \quad + \quad u = -16. & (8) \\
(8)\times 7 = & -7x + 7u = -112. & \\
(7) = & 7x - 13u = -38. & \\
\text{Adding,} & -6u = -150, & \\
 & u = 25. & \\
\text{Hence,} & z = 13, & \\
 & y = 37, \quad \text{and} \quad x = 41. &
\end{array}$$

22.

$$\begin{array}{lrr}
 & 4x - 3y + 2z - 5u = 0; & (1) \\
 & 3x - 2y + z - 2u = 12; & (2) \\
 & x - y - z + 3u = 4; & (3) \\
 & 2x - 3y + 3z - 4u = 2. & (4) \\
(1)\times 3 = & 12x - 9y + 6z - 15u = 0. & \\
(2)\times 4 = & 12x - 8y + 4z - 8u = 48. & \\
\text{Subtracting,} & -y + 2z - 7u = -48. & (5) \\
(1) = & 4x - 3y + 2z - 5u = 0. & \\
(3)\times 4 = & 4x - 4y - 4z + 12u = 16. & \\
\text{Subtracting,} & y + 6z - 17u = -16. & (6) \\
(1) = & 4x - 3y + 2z - 5u = 0. & \\
(4)\times 2 = & 4x - 6y + 6z - 8u = 4. & \\
\text{Subtracting,} & 3y - 4z + 3u = -4. & (7) \\
(5) = & -y + 2z - 7u = -48. & \\
(6) = & y + 6z - 17u = -16. & \\
\text{Adding,} & 8z - 24u = -64. & \\
\text{Or,} & z - 3u = -8. & (8) \\
(5)\times 3 = & -3y + 6z - 21u = -144. & \\
(7) = & 3y - 4z + 3u = -4. & \\
\text{Adding,} & 2z - 18u = -148. & (9)
\end{array}$$

$$
\begin{array}{ll}
(8)\times 2 = & 2z - 6u = -16. \\
(9) = & 2z - 18u = -148. \\
\text{Subtracting,} & 12u = 132, \\
& u = 11. \\
\text{Hence,} & z = 25, \\
& y = 21, \quad \text{and} \quad x = 17.
\end{array}
$$

23.

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62; \qquad (1)$$

$$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47; \qquad (2)$$

$$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38. \qquad (3)$$

$$
\begin{array}{lll}
\text{Or,} & 6x + 4y + 3z = 744; & (1) \\
& 20x + 15y + 12z = 2820; & (2) \\
& 15x + 12y + 10z = 2280. & (3) \\
(1)\times 4 = & 24x + 16y + 12z = 2976. & \\
(2) = & 20x + 15y + 12z = 2820. & \\
\text{Subtracting,} & 4x + y = 156. & (4) \\
(1)\times 10 = & 60x + 40y + 30z = 7440. & \\
(3)\times 3 = & 45x + 36y + 30z = 6840. & \\
\text{Subtracting,} & 15x + 4y = 600. & (5) \\
(4)\times 4 = & 16x + 4y = 624. & \\
(5) = & 15x + 4y = 600. & \\
\text{Subtracting,} & x = 24. & \\
\text{Hence,} & y = 60, \quad \text{and} \quad z = 120. &
\end{array}
$$

24.

$$\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=\frac{38}{5}; \qquad (1)$$

$$\frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=\frac{61}{6}; \qquad (2)$$

$$\frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=\frac{161}{10}. \qquad (3)$$

$$(1)\times 2 = \qquad \frac{6}{x}-\frac{8}{5y}+\frac{2}{z}=\frac{76}{5}.$$

$$(2) = \qquad \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=\frac{61}{6}.$$

$$\text{Subtracting,} \qquad \frac{17}{3x}-\frac{21}{10y}=\frac{151}{30}.$$

$$\text{Or,} \qquad \frac{170}{x}-\frac{63}{y}=151. \qquad (5)$$

$$(1)\times 4 = \qquad \frac{12}{x}-\frac{16}{5y}+\frac{4}{z}=\frac{152}{5}.$$

$$(3) = \qquad \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=\frac{161}{10}.$$

$$\text{Subtracting,} \qquad \frac{56}{5x}-\frac{27}{10y}=\frac{143}{10}.$$

$$\text{Or,} \qquad \frac{112}{x}-\frac{27}{y}=143. \qquad (6)$$

$$(5)\times 3 = \qquad \frac{510}{x}-\frac{189}{y}=453.$$

$$(6)\times 7 = \qquad \frac{784}{x}-\frac{189}{y}=1001.$$

$$\text{Subtracting,} \qquad \frac{274}{x}=548,$$

$$x=\tfrac{1}{2}.$$

Hence, $y=\tfrac{1}{3}$, and $z=\tfrac{1}{4}$.

25.

$$x+2z+3v+4t = 14; \qquad (1)$$
$$-x+3y+2z+5v+3t = \tfrac{129}{2}; \qquad (2)$$
$$x+6y-z+8v = 70; \qquad (3)$$
$$-x+y-z+4v-t = \tfrac{45}{2}; \qquad (4)$$
$$x-3y+3z-2v+4t = -24. \qquad (5)$$

$$(1) = \qquad x+2z+3v+4t = 14.$$

$$(2) = \qquad -x+2z+5v+3t+3y = \tfrac{129}{2}.$$

$$\text{Adding,} \qquad 4z+8v+7t+3y = \tfrac{157}{2}. \qquad (6)$$

$$\begin{array}{ll}
(1) = & x+2z+3v+4t = 14. \\
(3) = & x-z+8v +6y = 70. \\
\text{Subtracting,} & 3z-5v+4t-6y = -56. \qquad (7)
\end{array}$$

$$\begin{array}{ll}
(1) = & x+2z+3v+4t = 14. \\
(4) = & -x-z+4v-t+y = \tfrac{45}{2}. \\
\text{Adding,} & z+7v+3t+y = \tfrac{73}{2}. \qquad (8)
\end{array}$$

$$\begin{array}{ll}
(1) = & x+2z+3v+4t = 14. \\
(5) = & x+3z-2v+4t-3y = -24. \\
\text{Subtracting,} & -z+5v +3y = 38. \qquad (9)
\end{array}$$

$$\begin{array}{ll}
(8)\times 4 = & 4z+28v+12t+4y = 146. \\
(7)\times 3 = & 9z-15v+12t-18y = -168. \\
\text{Subtracting,} & -5z+43v +22y = 314. \qquad (10)
\end{array}$$

$$\begin{array}{ll}
(6)\times 3 = & 12z+24v+21t+9y = \tfrac{471}{2}. \\
(8)\times 7 = & 7z+49v+21t+7y = \tfrac{511}{2}. \\
\text{Subtracting,} & 5z-25v +2y = -20. \qquad (11)
\end{array}$$

$$\begin{array}{ll}
(9)\times 5 = & -5z+25v+15y = 190. \\
(10) = & -5z+43v+22y = 314. \\
\text{Subtracting,} & 18v+7y = 124. \qquad (12)
\end{array}$$

$$\begin{array}{ll}
(10) = & -5z+43v+22y = 314. \\
(11) = & 5z-25v+2y = -20. \\
\text{Adding,} & 18v+24y = 294. \qquad (13)
\end{array}$$

$$\begin{array}{ll}
(12) = & 18v+7y = 124. \\
(13) = & 18v+24y = 294. \\
\text{Subtracting,} & 17y = 170, \\
 & y = 10.
\end{array}$$

Hence, $x = -7,\ \ z = 7,\ \ v = 3,$ and $t = -\tfrac{1}{2}.$

26.

$$ax+by+cz = 8; \qquad (1)$$

$$ax+by-cz = 1; \qquad (2)$$

$$ax-by+cz = 1. \qquad (3)$$

$$\begin{array}{ll} (1) = & ax+by+cz = 8. \\ (2) = & \underline{ax+by-cz = 1.} \\ \text{Subtracting,} & 2cz = 2, \\ & z = \frac{1}{c}. \\ \text{Hence,} & y = \frac{1}{b}, \quad \text{and} \quad x = \frac{1}{a}. \end{array}$$

Art. 217.

1. Let x = the first number, and y = the second number.

$$\begin{array}{llll} \text{Then} & 3x+\frac{1}{2}y = 108; \ (1) & \frac{1}{4}x+\frac{1}{3}y = 32. & (2) \\ \text{Or,} & 6x+ y = 216; \ (1) & 3x+4y = 192. & (2) \end{array}$$

$$\begin{array}{ll} (2)\times 2 = & 6x+8y = 384. \\ (1) = & \underline{6x+ y = 216.} \\ \text{Subtracting,} & 7y = 168, \\ & y = 24, \text{ the second number.} \\ \text{Hence,} & x = 32, \text{ the first number.} \end{array}$$

2. Let $\frac{x}{y}$ = the fraction.

$$\begin{array}{llll} \text{Then} & \frac{x+3}{y} = \frac{1}{3}; \ (1) & \frac{x}{y-1} = \frac{1}{5}. & (2) \\ \text{Or,} & 3x-y = -9; \ (1) & 5x-y = -1. & (2) \end{array}$$

$$\begin{array}{ll} (1) = & 3x-y = -9. \\ (2) = & \underline{5x-y = -1.} \\ \text{Subtracting,} & 2x = 8, \\ & x = 4. \\ \text{Hence,} & y = 21, \\ \text{and} & \frac{x}{y} = \frac{4}{21}, \text{ the fraction.} \end{array}$$

3. Let x = the first number, and y = the second number.

$$\begin{array}{llll} \text{Then} & \frac{1}{2}x-\frac{1}{5}y = 19; \ (1) & \frac{1}{6}x-\frac{1}{4}y = 8. & (2) \\ \text{Or,} & 5x-2y = 190; \ (1) & 4x-3y = 96. & (2) \end{array}$$

$$\begin{aligned}
(1)\times 4 &= & 20x-\ 8y &= 760.\\
(2)\times 5 &= & 20x-15y &= 480.\\
\text{Subtracting,} & & 7y &= 280,\\
& & y &= 40, \text{ the second number.}\\
\text{Hence,} & & x &= 54, \text{ the first number.}
\end{aligned}$$

4. Let x = tens' digit, and y = units' digit.

Then $10x+y$ = the number.

Whence, $7(x+y) = 10x+y$; (1) $(10x+y)-27 = 10y+x$. (2)

Or, $3x-6y = 0$; (1) $9x-9y = 27$. (2)

$$\begin{aligned}
(1)\times 3 &= & 9x-18y &= 0.\\
(2) &= & 9x-\ 9y &= 27.\\
\text{Subtracting,} & & -9y &= -27,\\
& & y &= 3.\\
\text{Hence,} & & x &= 6,\\
\text{and} & & 10x+y &= 63, \text{ the number.}
\end{aligned}$$

5. Let x = tens' digit, and y = units' digit.

Then $10x+y$ = the number.

Whence, $y = 2x$; (1) $2(10x+y)-6 = 10y+x$. (2)

Or, $y-2x = 0$; (1) $19x-8y = 6$. (2)

$$\begin{aligned}
(1)\times 8 &= & 8y-16x &= 0.\\
(2) &= & -8y+19x &= 6.\\
\text{Adding,} & & 3x &= 6,\\
& & x &= 2.\\
\text{Hence,} & & y &= 4,\\
\text{and} & & 10x+y &= 24, \text{ the number.}
\end{aligned}$$

6. Let x = tens' digit, and y = units' digit.

Then $10x+y$ = the number.

Whence, $10x+y = 4(x+y)$; (1) $(10x+y)+18 = 10y+x$. (2)

Or, $2x-y = 0$; (1) $x-y = -2$. (2)

(1) =	$2x-y = 0.$
(2) =	$x-y = -2.$
Subtracting,	$x = 2.$
Hence,	$y = 4,$
and	$10x+y = 24,$ the number.

7. Let $\frac{x}{y}$ = the fraction.

Then $\frac{x+1}{y+1} = \frac{1}{2}$; (1) $\quad \frac{x-2}{y-3} = \frac{1}{3}$. (2)

Or, $2x-y = -1$; (1) $\quad 3x-y = 3.$ (2)

(1) =	$2x-y = -1.$
(2) =	$3x-y = 3.$
Subtracting,	$-x = -4,$
	$x = 4.$
Hence,	$y = 9,$
and	$\frac{x}{y} = \frac{4}{9}$, the fraction.

8. Let x = the first number, and y = the second number.

Then $\frac{2}{3}x+\frac{3}{5}y = y+4$; (1) $\quad 3x-8 = 2y.$ (2)

Or, $5x-3y = 30$; (1) $\quad 3x-2y = 8.$ (2)

(1) × 3 =	$15x-9y = 90.$
(2) × 5 =	$15x-10y = 40.$
Subtracting,	$y = 50,$ the second number.
Hence,	$x = 36,$ the first number.

9. Let x = A's wages for one day, y = B's wages for one day, and z = C's wages for one day.

Then
$$2x+6y+3z = \$28; \quad (1)$$
$$x+2y+4z = \$19; \quad (2)$$
$$5x+4y+2z = \$26. \quad (3)$$

(2) × 2 =	$2x+4y+8z = \$38.$
(1) =	$2x+6y+3z = \$28.$
Subtracting,	$-2y+5z = \$10.$ (4)

$$\begin{array}{lrl}
(2)\times 5 = & 5x+10y+20z = & \$95. \\
(3) = & 5x+\ \ 4y+\ \ 2z = & \$26. \\
\text{Subtracting,} & 6y+18z = & \$69. \\
\text{Or,} & 2y+\ \ 6z = & \$23. \quad (5)
\end{array}$$

$$\begin{array}{lrl}
(4) = & -2y+5z = & \$10. \\
(5) = & 2y+6z = & \$23. \\
\text{Adding,} & 11z = & \$33, \\
 & z = & \$3, \text{ C's wages for one day.} \\
\text{Hence,} & x = & \$2, \text{ A's wages for one day;} \\
\text{and} & y = & \$2\tfrac{1}{2}, \text{ B's wages for one day.}
\end{array}$$

10. Let $x =$ the first number, $y =$ the second number,
and $z =$ the third number.

$$\begin{array}{lrl}
\text{Then} & \tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{5}z = & 39; \quad (1) \\
 & \tfrac{1}{3}x+\tfrac{1}{4}y+\tfrac{1}{2}z = & 42; \quad (2) \\
 & \tfrac{1}{5}x+\tfrac{1}{6}y+\tfrac{1}{4}z = & 24. \quad (3)
\end{array}$$

$$\begin{array}{lrl}
\text{Or,} & 15x+10y+\ \ 6z = & 1170; \quad (1) \\
 & 4x+\ \ 3y+\ \ 6z = & 504; \quad (2) \\
 & 12x+10y+15z = & 1440. \quad (3)
\end{array}$$

$$\begin{array}{lrl}
(1) = & 15x+10y+6z = & 1170. \\
(2) = & 4x+\ \ 3y+6z = & 504. \\
\text{Subtracting,} & 11x+\ \ 7y \qquad = & 666. \quad (4)
\end{array}$$

$$\begin{array}{lrl}
(2)\times 5 = & 20x+15y+30z = & 2520. \\
(3)\times 2 = & 24x+20y+30z = & 2880. \\
\text{Subtracting,} & 4x+\ \ 5y \qquad = & 360. \quad (5)
\end{array}$$

$$\begin{array}{lrl}
(4)\times 4 = & 44x+28y = & 2664. \\
(5)\times 11 = & 44x+55y = & 3960. \\
\text{Subtracting,} & 27y = & 1296, \\
 & y = & 48, \text{ the second number.} \\
\text{Hence,} & x = & 30, \text{ the first number;} \\
\text{and} & z = & 40, \text{ the third number.}
\end{array}$$

11. Let $\frac{x}{y}$ = the fraction.

Then $\quad \frac{x+4}{y}=\frac{3}{4}; \quad (1) \qquad \frac{x}{y-2}=\frac{1}{2}. \quad (2)$

Or, $\quad 4x-3y=-16; \quad (1) \qquad 2x-y=-2. \quad (2)$

$$\begin{array}{ll} (2)\times 2 = & 4x-2y=-\ 4. \\ (1) = & \underline{4x-3y=-16.} \\ \text{Subtracting,} & y=\ \ 12. \\ \text{Hence,} & x=5, \\ \text{and} & \frac{x}{y}=\frac{5}{12}, \text{ the fraction.} \end{array}$$

12. Let x = No. persons, and y = No. dollars each received.

Then xy = No. dollars all received.

Whence, $\quad \frac{xy}{x+10}=y-2; \quad (1) \qquad \frac{xy}{x-5}=y+2. \quad (2)$

Or, $\quad x-5y=-10; \quad (1) \qquad -2x+5y=-10. \quad (2)$

$$\begin{array}{ll} (1) = & x-5y=-10. \\ (2) = & \underline{-2x+5y=-10.} \\ \text{Adding,} & -x \quad =-20, \\ & x=20, \text{ No. persons.} \\ \text{Hence,} & y=\ \ 6, \text{ No. dollars each received.} \end{array}$$

13. Let x = No. dollars A has, and y = No. dollars B has.

Then $\quad y-4=2(x+4); \quad (1) \qquad x-6=\frac{1}{3}(y+6). \quad (2)$

Or, $\quad -2x+y=12; \quad (1) \qquad 3x-y=24. \quad (2)$

$$\begin{array}{ll} (1) = & -2x+y=12. \\ (2) = & \underline{3x-y=24.} \\ \text{Adding,} & x \quad =36, \text{ No. dollars A has.} \\ \text{Hence,} & y=84, \text{ No. dollars B has.} \end{array}$$

14. Let $\quad x$ = No. dollars A can earn in one day,

y = No. dollars B can earn in one day,

and $\quad z$ = No. dollars C can earn in one day.

Then
$$10x+10y = 49; \quad (1)$$
$$10x+10z = 51; \quad (2)$$
$$4y+4z = 22. \quad (3)$$

$$\begin{array}{lrl} (1) = & 10x+10y &= 49. \\ (2) = & 10x +10z &= 51. \\ \hline \text{Subtracting,} & 10y-10z &= -2. \\ \text{Or,} & 5y-5z &= -1. \quad (4) \end{array}$$

$$\begin{array}{lrl} (3)\times 5 = & 20y+20z &= 110. \\ (4)\times 4 = & 20y-20z &= -4. \\ \hline \text{Adding,} & 40y &= 106, \end{array}$$

$y = 2.65$, No. dollars B can earn.

Hence, $x = 2.25$, No. dollars A can earn;

and $z = 2.85$, No. dollars C can earn.

15. Let $x =$ the first number, and $y =$ the second number.

Then $x+y = 230$; (1) $\frac{1}{11}x = \frac{1}{13}y+10.$ (2)

Or, $x+y = 230$; (1) $13x-11y = 1430.$ (2)

$$\begin{array}{lrl} (1)\times 11 = & 11x+11y &= 2530. \\ (2) = & 13x-11y &= 1430. \\ \hline \text{Adding,} & 24x &= 3960, \end{array}$$

$x = 165$, first number.

Hence, $y = 65$, second number.

16. Let $x =$ number of artillery, $y =$ number of cavalry,
and $z =$ number of infantry.

Then
$$x+y+z = 2744; \quad (1)$$
$$\tfrac{1}{3}z+245 = 3x+6y; \quad (2)$$
$$8(x+y) = z-98. \quad (3)$$

$$\begin{array}{lrl} (1) = & x+y+z &= 2744. \\ (2) = & 6x+12y-z &= 490. \\ \hline \text{Adding,} & 7x+13y &= 3234. \quad (4) \end{array}$$

$$
\begin{array}{ll}
(1) = & x + y + z = 2744. \\
(3) = & 8x + 8y - z = -98. \\
\text{Adding,} & 9x + 9y = 2674. \quad (5)
\end{array}
$$

$$
\begin{array}{ll}
(4) \times 9 = & 63x + 117y = 29106. \\
(5) \times 7 = & 63x + 63y = 18522. \\
\text{Subtracting,} & 54y = 10584, \\
 & y = 196, \text{ No. of cavalry.} \\
\text{Hence,} & x = 98, \text{ No. of artillery;} \\
\text{and} & z = 2450, \text{ No. of infantry.}
\end{array}
$$

17. Let x = No. dollars A has, and y = No. dollars B has.

Then $x + 145 = 2y$; (1) $y + 37 = \frac{4}{5}x$. (2)

Or, $x - 2y = -145$; (1) $-4x + 5y = -185$. (2)

$$
\begin{array}{ll}
(1) \times 4 = & 4x - 8y = -580. \\
(2) = & -4x + 5y = -185. \\
\text{Adding,} & -3y = -765, \\
 & y = 255, \text{ No. dollars B has.} \\
\text{Hence,} & x = 365, \text{ No. dollars A has.}
\end{array}
$$

18. Let x = No. bu. in the mixture @ \$1.10.

$75 - x$ = No. bu. in the mixture @ \$1.19.

Then $\$1.05\,(25) + \$1.10\,(x) + \$1.19\,(75 - x) = \$1.11\,(100)$.

Whence, $-\$.09x = -\$4.50$,

$x = 50$, No. bu. @ \$1.10.

Hence, $y = 25$, No. bu. @ \$1.19.

19. Let x = length in feet, and y = width in feet.

Then $(x + 6)(y + 2) = xy + 180$; (1)

$(x + 2)(y + 9) = xy + 324$. (2)

Or, $6y + 2x = 168$; (1) $2y + 9x = 306$. (2)

(2) × 3 = $\quad 6y+27x = 918.$

(1) = $\quad 6y+2x = 168.$

Subtracting, $\quad 25x = 750,$

$x = 30$ ft., length of parlor.

Hence, $\quad y = 18$ ft., width of parlor.

20. Let $x =$ No. min. in which A will fill it,

$y =$ No. min. in which B will fill it,

and $z =$ No. min. in which C will fill it.

Then
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{70}; \qquad (1)$$
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{84}; \qquad (2)$$
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{140}. \qquad (3)$$

(1) = $\quad \frac{1}{x}+\frac{1}{y}=\frac{1}{70}.$

(2) = $\quad \frac{1}{x}+\frac{1}{z}=\frac{1}{84}.$

Subtracting, $\quad \frac{1}{y}-\frac{1}{z}=\frac{1}{420}. \qquad (4)$

(3) = $\quad \frac{1}{y}+\frac{1}{z}=\frac{1}{140}.$

(4) = $\quad \frac{1}{y}-\frac{1}{z}=\frac{1}{420}.$

Subtracting, $\quad \frac{2}{z}=\frac{1}{210},$

$z = 420$, No. min. in which C can fill it.

Hence, $\quad y = 210$, No. min. in which B can fill it;

and $\quad x = 105$, No. min. in which A can fill it.

21. Let $x =$ No. dollars A has, and $y =$ No. dollars B has.

Then $\quad x+y = 13500; \quad (1) \qquad \frac{5}{8}x+\frac{3}{5}y = 8250. \quad (2)$

$$\begin{array}{lrl}
(2) = & 25x + 24y = & 330000. \\
(1) \times 25 = & 25x + 25y = & 337500. \\
 & y = & 7500, \text{ No. dollars B has.} \\
\text{Hence,} & x = & 6000, \text{ No. dollars A has.}
\end{array}$$

22. Let x = cost of one ox, y = cost of one sheep,
and z = cost of one lamb.

$$\begin{array}{lrll}
\text{Then} & 10x + 120y + 45z = & \$1280; & (1) \\
 & 3y + 7z = & x; & (2) \\
 & 2x + 6y + 8z = & \$128. & (3) \\
(1) = & 2x + 24y + 9z = & \$256. & \\
(3) = & 2x + 6y + 8z = & \$128. & \\
\text{Subtracting,} & 18y + z = & \$128. & (4) \\
(1) = & 2x + 24y + 9z = & \$256. & \\
(2) \times 2 = & -2x + 6y + 14z = & 0. & \\
\text{Adding,} & 30y + 23z = & \$256. & (5) \\
(4) \times 23 = & 414y + 23z = & \$2944. & \\
(5) = & 30y + 23z = & \$256. & \\
\text{Subtracting,} & 384y = & \$2688, & \\
 & y = & \$7, \text{ cost of a sheep.} & \\
\text{Hence,} & z = & \$2, \text{ cost of a lamb;} & \\
\text{and} & x = & \$35, \text{ cost of an ox.} &
\end{array}$$

23. Let x = A's daily wages, y = B's daily wages,
and z = C's daily wages.

$$\begin{array}{lrll}
\text{Then} & 10x + 4y + 3z = & \$29; & (1) \\
 & 9x + 8y + 6z = & \$36; & (2) \\
 & 7x + 6y + 4z = & \$27. & (3) \\
(1) \times 2 = & 20x + 8y + 6z = & \$58. & \\
(2) = & 9x + 8y + 6z = & \$36. & \\
\text{Subtracting,} & 11x = & \$22, & \\
 & x = & \$2, \text{ A's daily wages.} & \\
\text{Hence,} & y = & \$1.50, \text{ B's daily wages;} & \\
 & z = & \$1, \text{ C's daily wages} &
\end{array}$$

24. Let x = price of one horse, and y = price of one cow.

Then $9x+7y = \$2620$; (1) $3x+4y = \$1015$. (2)

$$
\begin{aligned}
(2)\times 3 &= & 9x+12y &= \$3045.\\
(1) &= & 9x+7y &= \$2620.\\
\text{Subtracting,} && 5y &= \$425,\\
&& y &= \$85, \text{ price of one cow.}\\
\text{Hence,} && x &= \$325, \text{ price of one horse.}
\end{aligned}
$$

25. Let x = No. dollars A had, y = No. dollars B had,

and z = No. dollars C had.

Then

$$
\begin{aligned}
x+\tfrac{1}{3}y+\tfrac{1}{2}z &= 5000; && (1)\\
\tfrac{1}{2}x+y+\tfrac{1}{4}z &= 5000; && (2)\\
\tfrac{1}{4}x+\tfrac{1}{6}y+z &= 5000. && (3)
\end{aligned}
$$

Or,

$$
\begin{aligned}
6x+2y+3z &= 30000; && (1)\\
2x+4y+z &= 20000; && (2)\\
3x+2y+12z &= 60000. && (3)
\end{aligned}
$$

$$
\begin{aligned}
(2)\times 3 &= & 6x+12y+3z &= 60000.\\
(1) &= & 6x+2y+3z &= 30000.\\
\text{Subtracting,} && 10y &= 30000,\\
&& y &= 3000, \text{ No. dollars B had.}\\
\text{Hence,} && x &= 2000, \text{ No. dollars A had;}\\
\text{and} && z &= 4000, \text{ No. dollars C had.}
\end{aligned}
$$

26. Let x = No. dollars A is worth, y = No. dollars B is worth,

and z = No. dollars C is worth.

Then

$$
\begin{aligned}
x+3y+3z &= 4700; && (1)\\
4x+y+4z &= 5800; && (2)\\
5x+5y+z &= 6300. && (3)
\end{aligned}
$$

$$
\begin{aligned}
(1)\times 4 &= & 4x+12y+12z &= 18800.\\
(2) &= & 4x+y+4z &= 5800.\\
\text{Subtracting,} && 11y+8z &= 13000. && (4)\\
(1)\times 5 &= & 5x+15y+15z &= 23500.\\
(3) &= & 5x+5y+z &= 6300.\\
\text{Subtracting,} && 10y+14z &= 17200. && (5)
\end{aligned}
$$

$$\begin{array}{lrl}
(4)\times 5 = & 55y+40z = & 65000. \\
(5)\times 11 = & 55y+77z = & 94600. \\
\text{Subtracting,} & 37z = & 29600, \\
 & z = & 800, \text{ No. dollars C owns.} \\
\text{Hence,} & y = & 600, \text{ No. dollars B owns;} \\
\text{and} & x = & 500, \text{ No. dollars A owns.}
\end{array}$$

27. Let x = No. of 3-cent pieces, y = No. of 10-cent pieces,

$310-y$ = No. of 25-cent pieces.

$$\begin{array}{lrll}
\text{Then} & .03x+.25(310-y) = & 43; & (1) \\
 & .10x+.03y = & 17.50. & (2)
\end{array}$$

$$\begin{array}{lrl}
(1) = & .03x-.15y = & -34.50. \\
(2)\times 5 = & .50x+.15y = & 87.50. \\
\text{Adding,} & .53x = & 53.00, \\
 & x = & 100, \text{ No. 3-cent pieces.} \\
\text{Hence,} & y = & 250, \text{ No. 10-cent pieces;} \\
\text{and} & z = & 60, \text{ No. 25-cent pieces.}
\end{array}$$

28. Let x = No. dollars A has, y = No. dollars B has,

and z = No. dollars C has.

$$\begin{array}{lrll}
\text{Then} & \frac{1}{6}y+\frac{1}{6}z = & \frac{1}{2}x-1500; & (1) \\
 & \frac{1}{4}y = & x-z; & (2) \\
 & x+z = & 3y-1000. & (3) \\
\text{Or,} & -3x+y+z = & -9000; & (1) \\
 & -4x+y+4z = & 0; & (2) \\
 & x-3y+z = & -1000. & (3)
\end{array}$$

$$\begin{array}{lrll}
(1) = & -3x+y+z = & -9000. & \\
(2) = & -4x+y+4z = & 0. & \\
\text{Subtracting,} & x-3z = & -9000. & (4) \\
(1)\times 3 = & -9x+3y+3z = & -27000. & \\
(3) = & x-3y+z = & -1000. & \\
\text{Adding,} & -8x+4z = & -28000. & (5)
\end{array}$$

$(4) \times 2 =$	$2x - 6z = -18000.$
$(5) \div 4 =$	$-2x + z = -7000.$
Adding,	$-5z = -25000,$
	$z = 5000,$ No. dollars C has.
Hence,	$x = 6000,$ No. dollars A has;
and	$y = 4000,$ No. dollars B has.

29. Let x = No. dollars A had at first, y = No. dollars B had at first, and z = No. dollars C had at first.

Then
$$y + 374 = z + 50; \quad (1)$$
$$x - 424 = y + 374; \quad (2)$$
$$y + z = x + 228. \quad (3)$$

Or,
$$y - z = -324; \quad (1)$$
$$x - y = 798; \quad (2)$$
$$-x + y + z = 228. \quad (3)$$

$(2) =$	$x - y = 798.$
$(3) =$	$-x + y + z = 228.$
Adding,	$z = 1026,$ No. dollars C had.
Hence,	$x = 1500,$ No. dollars A had;
and	$y = 702,$ No. dollars B had.

30. Let x = No. dollars A has, y = No. dollars B has, and z = No. dollars C has.

Then
$$x + y + z = 1820; \quad (1)$$
$$x + 200 = y + 160 - 200; \quad (2)$$
$$y + 70 = z - 70. \quad (3)$$

$(1) =$	$x + y + z = 1820.$	
$(2) =$	$x - y = -240.$	
Adding	$2x + z = 1580.$	(4)
$(2) =$	$x - y = -240.$	
$(3) =$	$y - z = -140.$	
Adding,	$x - z = -380.$	(5)

$$\begin{array}{lrl}
(4) = & 2x+z = & 1580. \\
(5) = & x-z = & -380. \\
\text{Adding,} & x = & 400, \text{ No. dollars A has.} \\
\text{Hence,} & y = & 640, \text{ No. dollars B has;} \\
\text{and} & z = & 780, \text{ No. dollars C has.}
\end{array}$$

31. Let $x =$ No. mi. from first to second angle,
$y =$ No. mi. from second to third angle,
and $z =$ No. mi. from first to third angle.

Then $y+z = 82$; (1) $x+y = 97$; (2) $x+z = 89$. (3)

$$\begin{array}{lrll}
(1) = & y+z = & 82. & \\
(2) = & x+y \quad = & 97. & \\
\text{Subtracting,} & x \quad -z = & 15. & (4) \\
(3) = & x+z = & 89. & \\
(4) = & x-z = & 15. & \\
\text{Adding,} & 2x \quad = & 104, & \\
& x = & 52, & \text{No. mi. from first to second.} \\
\text{Hence,} & y = & 45, & \text{No. mi. from second to third;} \\
\text{and} & z = & 37, & \text{No. mi. from first to third.}
\end{array}$$

32. Let $x =$ No. of dollars A has, $y =$ No. of dollars B has,
and $z =$ amount of bill.

Then $x+\frac{1}{8}y = z$; (1) $y+.50 = z$; (2) $\frac{1}{2}x+4 = z$. (3)

$$\begin{array}{lrll}
(1) = & 8x+y-8z = & 0. & \\
(2) = & y-\ z = & -.50. & \\
\text{Subtracting,} & 8x \quad -7z = & .50. & (4) \\
(3)\times 8 = & 8x-16z = & -64.00. & \\
(4) = & 8x-\ 7z = & .50. & \\
\text{Subtracting,} & -\ 9z = & -64.50, & \\
& z = & 7\frac{1}{6}, & \text{amt. of bill.} \\
\text{Hence,} & y = & 6\frac{2}{3}, & \text{No. dollars B has;} \\
\text{and} & x = & 6\frac{1}{3}, & \text{No. dollars A has.}
\end{array}$$

33. Let x = time in which first pipe can fill it,

y = time in which second pipe can fill it,

and z = time in which third pipe can fill it.

Then
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{100}; \qquad (1)$$
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{108}; \qquad (2)$$
$$\frac{1}{y}+\frac{1}{z}=\frac{11}{1350}. \qquad (3)$$

(1) = $\frac{1}{x}+\frac{1}{y}=\frac{1}{100}$.

(2) = $\frac{1}{x}+\frac{1}{z}=\frac{1}{108}$.

Subtracting, $\frac{1}{y}-\frac{1}{z}=\frac{1}{1350}$. (4)

(3) = $\frac{1}{y}+\frac{1}{z}=\frac{11}{1350}$.

(4) = $\frac{1}{y}-\frac{1}{z}=\frac{1}{1350}$.

Adding, $\frac{2}{y}=\frac{12}{1350}$,

y = 225 min., or $3\frac{3}{4}$ hrs., time for second to fill it.

Hence, x = 180 min., or 3 hrs., time for first to fill it;

and z = 270 min., or $4\frac{1}{2}$ hrs., time for third to fill it.

See Appendix.

34. Let x = cost per lb. of first kind, y = cost per lb. of second kind, and z = cost per lb. of third kind.

Then
$$3x+11y+4z = \$.22\times 18; \qquad (1)$$
$$9x+6y+3z = \$.21\times 18; \qquad (2)$$
$$5x+3y+6z = \$.20\times 14. \qquad (3)$$

(1) × 3 = $9x+33y+12z = \$11.88$.

(2) = $9x+6y+3z = \$3.78$.

Subtracting, $27y+9z = \$8.10$,

Or, $3y+z = \$.90$. (4)

$(1)\times 5 =$	$15x+55y+20z = \$19.80.$	
$(3)\times 3 =$	$15x+9y+18z = \$8.40.$	
Subtracting,	$46y+2z = \$11.40.$	
Or,	$23y+z = \$5.70.$	(5)

$(5) =$	$23y+z = \$5.70.$	
$(4) =$	$3y+z = \$.90.$	
Subtracting,	$20y = \$4.80,$	
	$y = \$.24$, cost per lb. of 2d kind.	
Hence,	$z = \$.18$, cost per lb. of 3d kind;	
	$x = \$.20$, cost per lb. of 1st kind.	

35. Let $x =$ hundreds' figure, $y =$ tens' figure, and $z =$ units' figure.

Then
$$x = y+z; \qquad (1)$$
$$\frac{100x+10y+z}{x+y+z} = 54; \qquad (2)$$
$$100x+10y+z-693 = 100z+10y+x. \qquad (3)$$

Or,
$$x-y-z = 0; \qquad (1)$$
$$40x-44y-53z = 0; \qquad (2)$$
$$x-z = 7. \qquad (3)$$

$(1) =$	$x-y-z = 0.$	
$(3) =$	$x \quad -z = 7.$	
Subtracting,	$y = 7$, tens' figure.	
Hence,	$x = 9$, hundreds' figure;	
	$z = 2$, units' figure;	
and	$100x+10y+z = 972$, the number.	

36. Let $x =$ No. dollars in first purse, $y =$ No. dollars in second purse, and $z =$ No. dollars in third purse.

Then
$$4(x-20) = y+20; \qquad (1)$$
$$z+(y+20-60) = 2[(x-20)+(y+20-60)]; \qquad (2)$$
$$z+(y+20-60)-40 = 2(x-20+40). \qquad (3)$$

Or,
$$4x-y = 100; \qquad (1)$$
$$-2x-y+z = -80; \qquad (2)$$
$$-2x+y+z = 120. \qquad (3)$$

$$
\begin{array}{lrl}
(2) = & -2x - y + z = & -80. \\
(3) = & -2x + y + z = & 120. \\
\text{Subtracting,} & -2y = & -200, \\
& y = & 100, \text{ No. dollars in second purse.} \\
\text{Hence,} & x = & 50, \text{ No. dollars in first purse;} \\
\text{and} & z = & 120, \text{ No. dollars in third purse.}
\end{array}
$$

37. Let x = rate per hour up hill, y = rate per hour down hill, and z = rate per hour on the level.

Then
$$\frac{24}{x} + \frac{36}{y} + \frac{12}{z} = 25; \qquad (1)$$
$$\frac{36}{x} + \frac{24}{y} + \frac{12}{z} = 28; \qquad (2)$$
$$\frac{12}{x} + \frac{36}{y} + \frac{24}{z} = 23. \qquad (3)$$

$$
\begin{array}{lrl}
(1) = & \frac{24}{x} + \frac{36}{y} + \frac{12}{z} = & 25. \\
(2) = & \frac{36}{x} + \frac{24}{y} + \frac{12}{z} = & 28. \\
\text{Subtracting,} & \frac{12}{x} - \frac{12}{y} = & 3. \qquad (4)
\end{array}
$$

$$
\begin{array}{lrl}
(2) \times 2 = & \frac{72}{x} + \frac{48}{y} + \frac{24}{z} = & 56. \\
(3) = & \frac{12}{x} + \frac{36}{y} + \frac{24}{z} = & 23. \\
\text{Subtracting,} & \frac{60}{x} + \frac{12}{y} = & 33. \qquad (5)
\end{array}
$$

$$
\begin{array}{lrl}
(4) = & \frac{12}{x} - \frac{12}{y} = & 3. \\
(5) = & \frac{60}{x} + \frac{12}{y} = & 33. \\
\text{Adding,} & \frac{72}{x} = & 36, \\
& x = & 2, \text{ rate up hill.} \\
\text{Hence,} & y = & 4, \text{ rate down hill;} \\
\text{and} & z = & 3, \text{ rate on the level.}
\end{array}
$$

38. Let x = No. days in which A can do the work,
and y = No. days in which B can do the work.

Then $\frac{1}{x}+\frac{1}{y}=\frac{1}{16}$; (1) $\frac{4}{x}+\frac{4}{y}+\frac{36}{y}=1$. (2)

$$(2) \div 4 = \quad \frac{1}{x}+\frac{10}{y}=\frac{1}{4}.$$

$$(1) = \quad \frac{1}{x}+\frac{1}{y}=\frac{1}{16}.$$

Subtracting, $\frac{9}{y}=\frac{3}{16}$,

$y = 48$, No. days B requires.

Hence, $x = 24$, No. days A requires.

39. Let x = the first number, y = the second number,
and z = the third number.

Then
$$x+y+z=26; \quad (1)$$
$$x+2y+4z=73; \quad (2)$$
$$3x+y+5z=88. \quad (3)$$

$$(1) = \quad x+y+z=26.$$
$$(2) = \quad x+2y+4z=73.$$
Subtracting, $y+3z=47$. (4)

$$(1)\times 3 = \quad 3x+3y+3z=78.$$
$$(3) = \quad 3x+y+5z=88.$$
Subtracting, $2y-2z=-10$. (5)

$$(5)\div 2 = \quad y-z=-5.$$
$$(4) = \quad y+3z=47.$$
Subtracting, $4z=52$,

$z = 13$, the third number.

Hence, $y = 8$, the second number;
and $x = 5$, the first number.

40. Let x = time in which A can do the work,
y = time in which B can do the work,
z = time in which C can do the work,

and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ = amount of work all can do in one day.

Then $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$; (1) $\frac{1}{x}+\frac{1}{z}=\frac{1}{15}$; (2) $\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$. (3)

(1) = $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$.

(2) = $\frac{1}{x}+\frac{1}{z}=\frac{1}{15}$.

Subtracting, $\frac{1}{y}-\frac{1}{z}=-\frac{1}{60}$. (4)

(3) = $\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$.

(4) = $\frac{1}{y}-\frac{1}{z}=\frac{1}{60}$.

Subtracting, $\frac{2}{z}=\frac{1}{30}$,

$z = 60$, time which C requires.

Hence, $y = 30$, time which B requires;

$x = 20$, time which A requires;

and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{10}$, amount of work all can do in one day.

If all can do $\frac{1}{10}$ in one day, they will do $\frac{10}{10}$ in ten days.

41. Let x = value of first horse, and y = value of second horse.

Then $x+\$50 = y+\$20$; (1)

$x+\$10 = \frac{y+\$50}{2}+\$30$. (2)

Or, $x-y = \$30$; (1) $2x-y = \$90$. (2)

(1) = $x-y = -\$30$.

(2) = $2x-y = \$90$.

Subtracting, $-x = -\$120$,

$x = \$120$, value of first horse.

Hence, $y = \$150$, value of second horse.

42. Let x = No. dollars each son receives,

y = No. dollars each daughter receives,

and z = No. dollars the mother receives.

Then
$$3x+4y+z = 29250; \quad (1)$$
$$x+y = 2z; \quad (2)$$
$$2x = y+z+2250. \quad (3)$$

$$\begin{aligned}
&(1) = & 3x+4y+\ z &= 29250.\\
&(2)\times 3 = & 3x+3y-6z &= \quad 0.\\
&\text{Subtracting,} & y+7z &= 29250. \quad (4)
\end{aligned}$$

$$\begin{aligned}
&(2)\times 2 = & 2x+2y-4z &= \quad 0.\\
&(3) = & 2x-\ y-\ z &= \ 2250.\\
&\text{Subtracting,} & 3y-3z &= -2250. \quad (5)
\end{aligned}$$

$$\begin{aligned}
&(5)\div 3 = & y-\ z &= -750.\\
&(4) = & y+7z &= 29250.\\
&\text{Subtracting,} & 8z &= 30000,
\end{aligned}$$

$z = 3750$, No. dollars the mother receives.

Hence, $y = 3000$, No. dollars each daughter receives;

and $x = 4500$, No. dollars each son receives.

Art. 223.

1. $(abx^2)^3 = abx^2 \times abx^2 \times abx^2 = a^3b^3x^6.$

2. $(-x^2y)^3 = -x^2y \times -x^2y \times -x^2y = -x^6y^3.$

3. $(-x^4yz^2)^4 = -x^4yz^2 \times -x^4yz^2 \times -x^4yz^2 \times -x^4yz^2 = x^{16}y^4z^8.$

4. $(3a^3x^2)^5 = 3a^3x^2 \times 3a^3x^2 \times 3a^3x^2 \times 3a^3x^2 \times 3a^3x^2 = 243a^{15}x^{10}.$

5. $(-2m^3n)^5 = -2m^3n \times -2m^3n \times -2m^3n \times -2m^3n \times -2m^3n$
$= -32m^{15}n^5.$

6. $(-3mn^2x)^4 = -3mn^2x \times -3mn^2x \times -3mn^2x \times -3mn^2x = 81m^4n^8x^4.$

7. $(5m^4n^3y^2)^2 = 5m^4n^3y^2 \times 5m^4n^3y^2 = 25m^8n^6y^4.$

8. $(-7a^3mn^2)^3 = -7a^3mn^2 \times -7a^3mn^2 \times -7a^3mn^2 = -343a^9m^3n^6.$

9. $(16a^4b^5m^9)^2 = 16a^4b^5m^9 \times 16a^4b^5m^9 = 256a^8b^{10}m^{18}.$

10. $(10x^2y^3z^4)^3 = 10x^2y^3z^4 \times 10x^2y^3z^4 \times 10x^2y^3z^4 = 1000x^6y^9z^{12}.$

11. $(-5a^2b^3c^8d^4)^4 = -5a^2b^3c^8d^4 \times -5a^2b^3c^8d^4 \times -5a^2b^3c^8d^4 \times -5a^2b^3c^8d^4$
$= 625a^8b^{12}c^{32}d^{16}.$

12. $(-4a^5m^3n^4x^2)^5 = -4a^5m^3n^4x^2 \times -4a^5m^3n^4x^2 \times -4a^5m^3n^4x^2 \times -4a^5m^3n^4x^2$
$\times -4a^5m^3n^4x^2 = -1024a^{25}m^{15}n^{20}x^{10}.$

13. $(\frac{1}{5}m^3)^2 = \frac{1}{25}m^6.$ $(\frac{1}{3}m^2n)^2 = \frac{1}{9}m^4n^2.$ $(4a^2b^2c^3)^2 = 16a^4b^4c^6.$

14. $(\frac{1}{3}a^2b)^3 = \frac{1}{27}a^6b^3.$ $(\frac{1}{4}x^3y^2z)^3 = \frac{1}{64}x^9y^6z^3.$ $(5am^4x^2)^3 = 125a^3m^{12}x^6.$

15. $(-10a^3bx^2)^2 = 100a^6b^2x^4.$ $(-15xyz^4)^2 = 225x^2y^2z^8.$

16. $(-\frac{5}{12}x^2yz^3)^3 = -\frac{5}{12}x^2yz^3 \times -\frac{5}{12}x^2yz^3 \times -\frac{5}{12}x^2yz^3 = -\frac{125}{1728}x^6y^3z^9.$
$(\frac{2}{11}m^4n^2y^3)^3 = \frac{2}{11}m^4n^2y^3 \times \frac{2}{11}m^4n^2y^3 \times \frac{2}{11}m^4n^2y^3 = \frac{8}{1331}m^{12}n^6y^9.$

17. $(\frac{1}{2}am^2n^2)^5 = \frac{1}{2}am^2n^2 \times \frac{1}{2}am^2n^2 \times \frac{1}{2}am^2n^2 \times \frac{1}{2}am^2n^2 \times \frac{1}{2}am^2n^2 = \frac{1}{32}a^5m^{10}n^{10}.$
$(-\frac{1}{3}x^2yz^3)^5 = -\frac{1}{3}x^2yz^3 \times -\frac{1}{3}x^2yz^3 \times -\frac{1}{3}x^2yz^3 \times -\frac{1}{3}x^2yz^3 \times -\frac{1}{3}x^2yz^3$
$= -\frac{1}{243}x^{10}y^5z^{15}.$

18. $(\frac{1}{4}p^2q)^2 = \frac{1}{16}p^4q^2.$ $(\frac{1}{5}ax^2y)^2 = \frac{1}{25}a^2x^4y^2.$ $(\frac{4}{5}am^2)^2 = \frac{16}{25}a^2m^4.$

19. $(\frac{2}{3}a^2m^3n)^4 = \frac{2}{3}a^2m^3n \times \frac{2}{3}a^2m^3n \times \frac{2}{3}a^2m^3n \times \frac{2}{3}a^2m^3n = \frac{16}{81}a^8m^{12}n^4.$
$(\frac{3}{4}ab^2c^2x)^4 = \frac{3}{4}ab^2c^2x \times \frac{3}{4}ab^2c^2x \times \frac{3}{4}ab^2c^2x \times \frac{3}{4}ab^2c^2x = \frac{81}{256}a^4b^8c^8x^4.$

20. $(\frac{4}{3}x^3yz^4)^3 = \frac{4}{3}x^3yz^4 \times \frac{4}{3}x^3yz^4 \times \frac{4}{3}x^3yz^4 = \frac{64}{27}x^9y^3z^{12}.$
$(\frac{9}{10}m^4n^3y^2)^3 = \frac{9}{10}m^4n^3y^2 \times \frac{9}{10}m^4n^3y^2 \times \frac{9}{10}m^4n^3y^2 = \frac{729}{1000}m^{12}n^9y^6.$

21. $\left(-\frac{2x^2}{3y}\right)^2 = \frac{4x^4}{9y^2}.$ $\left(\frac{3ax^2}{2y^3}\right)^2 = \frac{9a^2x^4}{4y^6}.$ $\left(-\frac{x^2y^2}{3az}\right)^2 = \frac{x^4y^4}{9a^2z^2}.$

22. $\left(\frac{4a^2x^4}{5y^2z}\right)^3 = \frac{4a^2x^4}{5y^2z} \times \frac{4a^2x^4}{5y^2z} \times \frac{4a^2x^4}{5y^2z} = \frac{64a^6x^{12}}{125y^6z^3}.$

$\left(-\frac{3a^3xy}{8m^2n^3}\right)^3 = -\frac{3a^3xy}{8m^2n^3} \times -\frac{3a^3xy}{8m^2n^3} \times -\frac{3a^3xy}{8m^2n^3} = -\frac{27a^9x^3y^3}{512m^6n^9}.$

23. $\left(\frac{-3a^2b^4c}{-5xy^2z^2}\right)^4 = \frac{-3a^2b^4c}{-5xy^2z^2} \times \frac{-3a^2b^4c}{-5xy^2z^2} \times \frac{-3a^2b^4c}{-5xy^2z^2} \times \frac{-3a^2b^4c}{-5xy^2z^2}$
$= \frac{81a^8b^{16}c^4}{625x^4y^8z^8}.$

$\left(\frac{2m^3n^2}{-3x^2y^3z^3}\right)^4 = \frac{2m^3n^2}{-3x^2y^3z^3} \times \frac{2m^3n^2}{-3x^2y^3z^3} \times \frac{2m^3n^2}{-3x^2y^3z^3} \times \frac{2m^3n^2}{-3x^2y^3z^3}$
$= \frac{32m^{12}n^8}{81x^8y^{12}z^{12}}.$

Art. 229.

1. $(a+b+c)^3=[(a+b)+c]^3=(a+b)^3+[3(a+b)^2\times c]+[3(a+b)\times c^2]+c^3$
$=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$

2. $(x-y+z)^3=[(x-y)+z]^3=(x-y)^3+[3(x-y)^2\times z]+[3(x-y)\times z^2]+z^3$
$=x^3-3x^2y+3xy^2-y^3+3x^2z-6xyz+3y^2z+3xz^2-3yz^2+z^3.$

3. $(m-n-r)^3=[(m-n)-r]^3$
$=(m-n)^3+[3(m-n)^2\times -r]+[3(m-n)\times -r^2]-r^3$
$=m^3-3m^2n+3mn^2-n^3-3m^2r+6mnr-3n^2r+3mr^2-3nr^2-r^3$

4. $(a+2b+c)^3=[(a+2b)+c]^3$
$=(a+2b)^3+[3(a+2b)^2\times c]+[3(a+2b)\times c^2]+c^3$
$=a^3+6a^2b+12ab^2+8b^3+3a^2c+12abc+12b^2c+3ac^2+6bc^2+c^3.$

5. $(x-2y+z)^3=[(x-2y)+z]^3$
$=(x-2y)^3+[3(x-2y)^2\times z]+[3(x-2y)\times z^2]+z^3$
$=x^3-6x^2y+12xy^2-8y^3+3x^2z-12xyz+12y^2z+3xz^2-6yz^2+z^3.$

6. $(x+2y+3z)^3=[(x+2y)+3z]^3$
$=(x+2y)^3+[3(x+2y)^2\times 3z]+[3(x+2y)\times(3z)^2]+(3z)^3$
$=x^3+6x^2y+12xy^2+8y^3+9x^2z+36xyz+36y^2z+27xz^2+54yz^2$
$+27z^3.$

7. $(a-2b-3c)^3=[(a-2b)-3c]^3$
$=(a-2b)^3+[3(a-2b)^2\times -3c]+[3(a-2b)\times(-3c)^2]+(-3c)^3$
$=a^3-6a^2b+12ab^2-8b^3-9a^2c+36abc-36b^2c+27ac^2-54bc^2-27c^3.$

8. $(2a-m+2n)^3=[(2a-m)+2n)^3$
$=(2a-m)^3+[3(2a-m)^2\times 2n]+[3(2a-m)\times(2n)^2]+(2n)^3$
$=8a^3-12a^2m+6am^2-m^3+24a^2n-24amn+6m^2n+24an^2$
$-12mn^2+8n^3.$

9. $(2p-3q-5r)^3=[(2p-3q)-5r]^3$
$=(2p-3q)^3+[3(2p-q)^2\times -5r]+3(2p-q)\times(-5r)^2]+(-5r)^3$
$=8p^3-36p^2q+54pq^2-27q^3-60p^2r+180pqr-135q^2r+150pr^2$
$-225qr^2-125r^3.$

10. $(a-5y+z)^3=[(a-5y)+z]^3$
$=(a-5y)^3+[3(a-5y)^2\times z]+[3(a-5y)\times z^2]+z^3$
$=a^3-15a^2y+75ay^2-125y^3+3a^2z-30ayz+75y^2z+3az^2-15yz^2$
$+z^3.$

11. $(a^2+x^2-y^2)^3 = [(a^2+x^2)-y^2]^3$
$= (a^2+x^2)^3+[3(a^2+x^2)^2 \times -y^2]+[3(a^2+x^2)\times(-y^2)^2]+(-y^2)^3$
$= a^6+3a^4x^2+3a^2x^4+x^6-3a^4y^2-6a^2x^2y^2-3x^4y^2+3a^2y^4+3x^2y^4-y^6.$

12. $(3x^2+y^2+2z)^3 = [(3x^2+y^2)+2z]^3$
$= (3x^2+y^2)^3+[3(3x^2+y^2)^2\times 2z]+[3(3x^2+y^2)\times(2z)^2]+(2z)^3$
$= 27x^6+27x^4y^2+9x^2y^4+y^6+54x^4z+36x^2y^2z+6y^4z+36x^2z^2+12y^2z^2$
$+8z^3.$

13. $(5x-2x^2+x^3)^3 = [(5x-2x^2)+x^3]^3$
$= (5x-2x^2)^3+[3(5x-2x^2)^2\times x^3]+[3(5x-2x^2)\times x^6]+x^9$
$= 125x^3-150x^4+135x^5-68x^6+27x^7-6x^8+x^9.$

14. $(a^2+ab+b^2)^3 = [(a^2+ab)+b^2]^3$
$= (a^2+ab)^3+[3(a^2+ab)^2\times b^2]+[3(a^2+ab)\times b^4]+b^6$
$= a^6+3a^5b+6a^4b^2+7a^3b^3+6a^2b^4+3ab^5+b^6.$

15. $(x^2-xy+y^2)^3 = [(x^2-xy)+y^2]^3$
$= (x^2-xy)^3+[3(x^2-xy)^2+y^2]+[3(x^2-xy)\times y^4]+y^6$
$= x^6-3x^5y+6x^4y^2-7x^3y^3+6x^2y^4-3xy^5+y^6.$

16. $(x^2+2x+3)^3 = [(x^2+2x)+3]^3$
$= (x^2+2x)^3+[3(x^2+2x)^2\times 3]+[3(x+2x)\times 9]+27$
$= x^6+6x^5+21x^4+44x^3+63x^2+54x+27.$

17. $(2x^2+3xy+y^2)^3 = [(2x^2+3xy)+y^2]^3$
$= (2x^2+3xy)^3+[3(2x^2+3xy)^2\times y^2]+[3(2x^2+3xy)\times y^4]+y^6$
$= 8x^6+36x^5y+66x^4y^2+63x^3y^3+33x^2y^4+9xy^5+y^6.$

18. $(3a^2-2ay+5y^2)^3 = [(3a^2-2ay)+5y^2]^3$
$= (3a^2-2ay)^3+[3(3a^2-2ay)^2\times 5y^2]+[3(3a^2-2ay)\times 25y^4]+125y^6$
$= 27a^6-54a^5y+171a^4y^2-188a^3y^3+285a^2y^4-150ay^5+125y^6.$

19. $(m^2-5mn+3n^2)^3 = [(m^2-5mn)+3n^2]^3$
$= (m^2-5mn)^3+[3(m^2-5mn)^2\times 3n^2]+[3(m^2-5mn)\times 9n^4]+27n^6$
$= m^6-15m^5n+84m^4n^2-215m^3n^3+252m^2n^4-135mn^5+27n^6.$

20. $(3a^4+5a^2y^2-2y^4)^3 = [3a^4+5a^2y^2)-2y^4]^3$
$= (3a^4+5a^2y^2)^3+[3(3a^4+5a^2y^2)^2\times -2y^4]$
$+[3(3a^4+5a^2y^2)\times(-2y^4)^2]+(-2y^4)^3$
$= 27a^{12}+135a^{10}y^2+171a^8y^4-55a^6y^6-114a^4y^8+60a^2y^{10}-8y^{12}.$

21. $(x^4-x^2y^2+y^4)^3 = [(x^4-x^2y^2)+y^4]^3$
$= (x^4-x^2y^2)^3+[3(x^4-x^2y^2)^2\times y^4]+[3(x^4-x^2y^2)\times y^8]+y^{12}$
$= x^{12}-3x^{10}y^2+6x^8y^4-7x^6y^6+6x^4y^8-3x^2y^{10}+y^{12}.$

22. $(1+p^2-p^4)^3 = [(1+p^2)-p^4]^3$
$= (1+p^2)^3+[3(1+p^2)^2\times -p^4]+[3(1+p^2)\times -p^8]-p^{12}$
$= 1+3p^2-5p^6+3p^{10}-p^{12}.$

23. $(2x^2-3xy+y^2)^3 = [(2x^2-3xy)+y^2]^3$
$= (2x^2-3xy)^3+[3(2x^2-3xy)^2\times y^2]+[3(2x^2-3xy)\times y^4]+y^6$
$= 8x^6-36x^5y+66x^4y^2-63x^3y^3+33x^2y^4-9xy^5+y^6.$

24. $(x^4-3x^2y-y^2)^3 = [(x^4-3x^2y)-y^2]^3$
$= (x^4-3x^2y)^3+[3(x^4-3x^2y)^2\times -y^2]+[3(x^4-3x^2y)\times -y^4]-y^6$
$= x^{12}-9x^{10}y+24x^8y^2-9x^6y^3-24x^4y^4-9x^2y^5-y^6.$

25. $(x^2-2xy-y^2)^3 = [(x^2-2xy)-y^2]^3$
$= (x^2-2xy)^3+[3(x^2-2xy)^2\times -y^2]+[3(x^2-2xy)\times -y^4]-y^6$
$= x^6-6x^5y+9x^4y^2+4x^3y^3-9x^2y^4-6xy^5-y^6.$

26. $(x^3-2x^2y^2+y^3)^3 = [(x^3-2x^2y^2)+y^3]^3$
$= (x^3-2x^2y^2)^3+[3(x^3-2x^2y^2)^2\times y^3]+[3(x^3-2x^2y^2)\times y^6]+y^9$
$= x^9-6x^8y^2+12x^7y^4-8x^6y^6+3x^6y^3-12x^5y^5+12x^4y^7+3x^3y^6-6x^2y^8$
$+y^9.$

27. $(x^2-5x+6)^3 = [(x^2-5x)+6]^3$
$= (x^2-5x)^3+[3(x^2-5x)^2\times 6]+[3(x^2-5x)\times 36]+216$
$= x^6-15x^5+93x^4-305x^3+558x^2-540x+216.$

28. $(x^4-9x^2y+8y^2)^3 = [(x^4-9x^2y)+8y^2]^3$
$= (x^4-9x^2y)^3+[3(x^4-9x^2y)^2\times 8y^2]+[3(x^4-9x^2y)\times 64y^4]+512y^6$
$= x^{12}-27x^{10}y+267x^8y^2-1161x^6y^3+2136x^4y^4-1728x^2y^5+512y^6.$

Art. 232.

1. $(a+x)^2 = a^2+2(a\times x)+x^2 = a^2+2ax+x^2.$

2. $(m-n)^2 = m^2-2(m\times n)+n^2 = m^2-2mn+n^2.$

3. $(a+b)^3 = a^3+3(a^2\times b)+3(a\times b^2)+b^3 = a^3+3a^2b+3ab^2+b^3.$

4. $(p+q)^4 = p^4+4(p^3\times q)+6(p^2\times q^2)+4(p\times q^3)+q^4$
$= p^4+4p^3q+6p^2q^2+4pq^3+q^4.$

5. $(2a+x)^2 = 4a^2+2(2a\times x)+x^2 = 4a^2+4ax+x^2.$

6. $(x-2y)^3 = x^3-3(x^2 \times 2y)+3(x \times 4y^2)-6y^3 = x^3-6x^2y+12xy^2-8y^3.$

7. $(2a+3m)^3 = 8a^3+3(4a^2 \times 3m)+3(2a \times 9m^2)+27m^3$
$= 8a^3+36a^2m+54am^2+27m^3.$

8. $(3x+5)^3 = 27x^3+3(9x^2 \times 5)+3(3x \times 25)+125$
$= 27x^3+135x^2+225x+125.$

9. $(2x+5y)^3 = 8x^3+3(4x^2 \times 5y)+3(2x \times 25y^2)+125y^3$
$= 8x^3+60x^2y+150xy^2+125y^3.$

10. $(5x+1)^4 = 625x^4+4(125x^3 \times 1)+6(25x^2 \times 1)+4(5x \times 1)+1$
$= 625x^4+500x^3+150x^2+20x+1.$

11. $(1+x)^4 = 1+4(1 \times x)+6(1 \times x^2)+4(1 \times x^3)+x^4$
$= 1+4x+6x^2+4x^3+x^4.$

12. $(a+b)^5 = a^5+5(a^4 \times b)+10(a^3 \times b^2)+10(a^2 \times b^3)+5(a \times b^4)+b^5$
$= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$

13. $(a-b)^5 = a^5-5(a^4 \times b)+10(a^3 \times b^2)-10(a^2 \times b^3)+5(a \times b^4)-b^5$
$= a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$

14. $(2x+y)^5 = 32x^5+5(16x^4 \times y)+10(8x^3 \times y^2)+10(4x^2 \times y^3)+5(2x \times y^4)$
$+y^5$
$= 32x^5+80x^4y+80x^3y^2+40x^2y^3+10xy^4+y^5.$

15. $(x-2y)^5 = x^5-5(x^4 \times 2y)+10(x^3 \times 4y^2)-10(x^2 \times 8y^3)+5(x \times 16y^4)-32y^5$
$= x^5-10x^4y+40x^3y^2-80x^2y^3+80xy^4-32y^5.$

16. $(3x+y)^5 = 243x^5+5(81x^4 \times y)+10(27x^3 \times y^2)+10(9x^2 \times y^3)+5(3x \times y^4)$
$+y^5$
$= 243x^5+405x^4y+270x^3y^2+90x^2y^3+15xy^4+y^5.$

17. $(m+5n)^5 = m^5+5(m^4 \times 5n)+10(m^3 \times 25n^2)+10(m^2 \times 125n^3)$
$+5(m \times 625n^4)+3125n^5$
$= m^5+25m^4n+250m^3n^2+1250m^2n^3+3125mn^4+3125n^5.$

18. $(2p-3q)^5 = 32p^5-5(16p^4 \times 3q)+10(8p^3 \times 9q^2)-10(4p^2 \times 27q^3)$
$+5(2p \times 81q^4)-243q^5$
$= 32p^5-240p^4q+720p^3q^2-1080p^2q^3+810pq^4-243q^5.$

19. $(a+2b)^4 = a^4+4(a^3 \times 2b)+6(a^2 \times 4b^2)+4(a \times 8b^3)+16b^4$
$= a^4+8a^3b+24a^2b^2+32ab^3+16b^4.$

20. $(2x+y)^4 = 16x^4+4(8x^3 \times y)+6(4x^2 \times y^2)+4(2x \times y^3)+y^4$
$= 16x^4+32x^3y+24x^2y^2+8xy^3+y^4.$

21. $(\frac{3}{4}x+y)^3 = \frac{27}{64}x^3+3(\frac{9}{16}x^2\times y)+3(\frac{3}{4}x\times y^2)+y^3$
$= \frac{27}{64}x^3+\frac{27}{16}x^2y+\frac{9}{4}xy^2+y^3$

22. $(x-\frac{1}{2}y)^3 = x^3-3(x^2\times\frac{1}{2}y)+3(x\times\frac{1}{4}y^2)-\frac{1}{8}y^3$
$= x^3-\frac{3}{2}x^2y+\frac{3}{4}xy^2-\frac{1}{8}y^3.$

23. $(\frac{2}{3}a-\frac{3}{4}x)^3 = \frac{8}{27}a^3-3(\frac{4}{9}a^2\times\frac{3}{4}x)+3(\frac{2}{3}a\times\frac{9}{16}x^2)-\frac{27}{64}x^3$
$= \frac{8}{27}a^3-a^2x+\frac{9}{8}ax^2-\frac{27}{64}x^3.$

24. $(a^2+b^2)^3 = a^6+3(a^4\times b^2)+3(a^2\times b^4)+b^6 = a^6+3a^4b^2+3a^2b^4+b^6.$

25. $(x+y^2)^3 = x^3+3(x^2\times y^2)+3(x\times y^4)+y^6 = x^3+3x^2y^2+3xy^4+y^6.$

26. $(\frac{1}{2}m-n^2)^4 = \frac{1}{16}m^4-4(\frac{1}{8}m^3\times n^2)+6(\frac{1}{4}m^2\times n^4)-4(\frac{1}{2}m\times n^6)+n^8$
$= \frac{1}{16}m^4-\frac{1}{2}m^3n^2+\frac{3}{2}m^2n^4-2mn^6+n^8.$

27. $(ax+1)^5 = a^5x^5+5(a^4x^4\times 1)+10(a^3x^3\times 1)+10(a^2x^2\times 1)+5(ax\times 1)+1$
$= a^5x^5+5a^4x^4+10a^3x^3+10a^2x^2+5ax+1.$

28. $(3a+xy)^3 = 27a^3+3(9a^2\times xy)+3(3a\times x^2y^2)+x^3y^3$
$= 27a^3+27a^2xy+9ax^2y^2+x^3y^3.$

29. $(\frac{3}{2}x+\frac{2}{3}y)^4 = \frac{81}{16}x^4+4(\frac{27}{8}x^3\times\frac{2}{3}y)+6(\frac{9}{4}x^2\times\frac{4}{9}y^2)+4(\frac{3}{2}x\times\frac{8}{27}y^3)+\frac{16}{81}y^4$
$= \frac{81}{16}x^4+9x^3y+6x^2y^2+\frac{16}{9}xy^3+\frac{16}{81}y^4.$

30. $(\frac{2}{3}x-\frac{3}{4}y)^4 = \frac{16}{81}x^4-4(\frac{8}{27}x^3\times\frac{3}{4}y)+6(\frac{4}{9}x^2\times\frac{9}{16}y^2)-4(\frac{2}{3}x\times\frac{27}{64}y^3)+\frac{81}{256}y^4$
$= \frac{16}{81}x^4-\frac{8}{9}x^3y+\frac{3}{2}x^2y^2-\frac{9}{8}xy^3+\frac{81}{256}y^4.$

Art. 244.

1. $\sqrt{25a^2b^4} = \sqrt{5a^{\frac{2}{2}}b^{\frac{4}{2}}\times 5a^{\frac{2}{2}}b^{\frac{4}{2}}} = \pm\, 5ab^2.$

2. $\sqrt{36x^4y^6} = \sqrt{6x^{\frac{4}{2}}y^{\frac{6}{2}}\times 6x^{\frac{4}{2}}y^{\frac{6}{2}}} = \pm\, 6x^2y^3.$

3. $\sqrt{144m^4n^6} = \sqrt{12m^{\frac{4}{2}}n^{\frac{6}{2}}\times 12m^{\frac{4}{2}}n^{\frac{6}{2}}} = \pm\, 12m^2n^3.$

4. $\sqrt{81a^4b^6c^{12}} = \sqrt{9a^{\frac{4}{2}}b^{\frac{6}{2}}c^{\frac{12}{2}}\times 9a^{\frac{4}{2}}b^{\frac{6}{2}}c^{\frac{12}{2}}} = \pm\, 9a^2b^3c^6.$

5. $\sqrt[3]{125x^3y^6z^9} = +5x^{\frac{3}{3}}y^{\frac{6}{3}}z^{\frac{9}{3}} = 5xy^2z^3.$

6. $\sqrt[4]{81x^4y^8z^{12}} = \pm\, 3x^{\frac{4}{4}}y^{\frac{8}{4}}z^{\frac{12}{4}} = \pm\, 3xy^2z^3.$

7. $\sqrt[5]{32p^5q^{10}r^{15}} = +2p^{\frac{5}{5}}q^{\frac{10}{5}}r^{\frac{15}{5}} = 2pq^2r^3.$

8. $\sqrt[4]{256a^8b^4c^{12}} = \pm\, 4a^{\frac{8}{4}}b^{\frac{4}{4}}c^{\frac{12}{4}} = \pm\, 4a^2bc^3.$

9. $\sqrt[3]{-27m^6n^3} = -3m^{\frac{6}{3}}n^{\frac{3}{3}} = -3m^2n.$

10. $\sqrt[5]{-32a^5m^{10}x^{15}} = -2a^{\frac{5}{5}}m^{\frac{10}{5}}x^{\frac{15}{5}} = -2am^2x^3.$

11. $\sqrt[4]{625a^8b^{12}c^4} = \pm 5a^{\frac{8}{4}}b^{\frac{12}{4}}c^{\frac{4}{4}} = \pm 5a^2b^3c.$

12. $\sqrt[5]{243a^{10}b^{15}c^{20}} = +3a^{\frac{10}{5}}b^{\frac{15}{5}}c^{\frac{20}{5}} = 3a^2b^3c^4.$

13. $\sqrt[3]{512x^6y^3z^{15}} = +8x^{\frac{6}{3}}y^{\frac{3}{3}}z^{\frac{15}{3}} = 8x^2yz^5.$

14. $\sqrt[3]{1331a^3b^6x^9} = +11a^{\frac{3}{3}}b^{\frac{6}{3}}x^{\frac{9}{3}} = 11ab^2x^3.$

15. $\sqrt[5]{-243a^5x^{10}y^{20}} = -3a^{\frac{5}{5}}x^{\frac{10}{5}}y^{\frac{20}{5}} = -3ax^2y^4.$

16. $\sqrt[3]{-125m^6n^9p^{12}} = -5m^{\frac{6}{3}}n^{\frac{9}{3}}p^{\frac{12}{3}} = -5m^2n^3p^4.$

17. $\sqrt{\frac{64}{81}a^2b^4c^6} = \pm \frac{8}{9}a^{\frac{2}{2}}b^{\frac{4}{2}}c^{\frac{6}{2}} = \pm \frac{8}{9}ab^2c^3.$

18. $\sqrt[3]{\frac{27}{64}a^3b^6c^9} = +\frac{3}{4}a^{\frac{3}{3}}b^{\frac{6}{3}}c^{\frac{9}{3}} = \frac{3}{4}ab^2c^3.$

19. $\sqrt[3]{-\frac{64a^3x^6y^3}{125m^3n^{12}r^6}} = -\frac{4a^{\frac{3}{3}}x^{\frac{6}{3}}y^{\frac{3}{3}}}{5m^{\frac{3}{3}}n^{\frac{12}{3}}r^{\frac{6}{3}}} = -\frac{4ax^2y}{5mn^4r^2}.$

20 $\sqrt[3]{-\frac{125}{216}a^6b^3x^{12}} = -\frac{5}{6}a^{\frac{6}{3}}b^{\frac{3}{3}}x^{\frac{12}{3}} = -\frac{5}{6}a^2bx^4.$

21. $\sqrt[3]{\frac{343}{1000}a^3b^{12}c^9} = \frac{7}{10}a^{\frac{3}{3}}b^{\frac{12}{3}}c^{\frac{9}{3}} = \frac{7}{10}ab^4c^3.$

22. $\sqrt[4]{\frac{16}{625}x^8y^{12}z^{16}} = \pm \frac{2}{5}x^{\frac{8}{4}}y^{\frac{12}{4}}z^{\frac{16}{4}} = \pm \frac{2}{5}x^2y^3z^4.$

23. $\sqrt[5]{\frac{32}{243}m^{20}n^{10}r^{15}} = \frac{2}{3}m^{\frac{20}{5}}n^{\frac{10}{5}}r^{\frac{15}{5}} = \frac{2}{3}m^4n^2r^3.$

24. $\sqrt[3]{\frac{216a^3b^6c^{15}}{125x^6y^9z^{12}}} = \frac{6a^{\frac{3}{3}}b^{\frac{6}{3}}c^{\frac{15}{3}}}{5x^{\frac{6}{3}}y^{\frac{9}{3}}z^{\frac{12}{3}}} = \frac{6ab^2c^5}{5x^2y^3z^4}.$

Art. 246.

1.

$$\begin{array}{r|l|l}
 & a^2+2ab+b^2 & \underline{a+b} \\
 & \underline{a^2} & \\
\underline{2a+b} & 2ab+b^2 & \\
 & \underline{2ab+b^2} &
\end{array}$$

2.
$$\begin{array}{r|l|l} & a^2+6a+9 & a+3 \\ & a^2 & \\ \hline 2a+3 & 6a+9 & \\ & 6a+9 & \\ \hline \end{array}$$

3.
$$\begin{array}{r|l|l} & x^2-2xy+y^2 & x-y \\ & x^2 & \\ \hline 2x-y & -2xy+y^2 & \\ & -2xy+y^2 & \\ \hline \end{array}$$

4.
$$\begin{array}{r|l|l} & m^2-2mn+n^2 & m-n \\ & m^2 & \\ \hline 2m-n & -2mn+n^2 & \\ & -2mn+n^2 & \\ \hline \end{array}$$

5.
$$\begin{array}{r|l|l} & a^2+4ab+4b^2 & a+2b \\ & a^2 & \\ \hline 2a+2b & 4ab+4b^2 & \\ & 4ab+4b^2 & \\ \hline \end{array}$$

6.
$$\begin{array}{r|l|l} & m^2+6mn+9n^2 & m+3n \\ & m^2 & \\ \hline 2m+3n & 6mn+9n^2 & \\ & 6mn+9n^2 & \\ \hline \end{array}$$

7.
$$\begin{array}{r|l|l} & a^2-8ab+16b^2 & a-4b \\ & a^2 & \\ \hline 2a-4b & -8ab+16b^2 & \\ & -8ab+16b^2 & \\ \hline \end{array}$$

8.
$$\begin{array}{r|l|l} & x^2+14xy+49y^2 & x+7y \\ & x^2 & \\ \hline 2x+7y & 14xy+49y^2 & \\ & 14xy+49y^2 & \\ \hline \end{array}$$

9.
$$\begin{array}{r|l|l} & a^2x^2+2ax+1 & ax+1 \\ & a^2x^2 & \\ \hline 2ax+1 & 2ax+1 & \\ & 2ax+1 & \\ \hline \end{array}$$

10.
$$\begin{array}{r|l|l} & a^2b^2-6ab+9 & ab-3 \\ & a^2b^2 & \\ \hline 2ab-3 & -6ab+9 & \\ & -6ab+9 & \\ \hline \end{array}$$

11.
$$\begin{array}{r|l}
 & 1+6mn+9m^2n^2 \quad \lfloor 1+3mn \\
 & 1 \\ \hline
2+3mn & 6mn+9m^2n^2 \\
 & 6mn+9m^2n^2 \\ \hline
\end{array}$$

12.
$$\begin{array}{r|l}
 & 4x^2-12xy+9y^2 \quad \lfloor 2x-3y \\
 & 4x^2 \\ \hline
4x-3y & -12xy+9y^2 \\
 & -12xy+9y^2 \\ \hline
\end{array}$$

13.
$$\begin{array}{r|l}
 & 100m^4-180m^2n^2+81n^4 \quad \lfloor 10m^2-9n^2 \\
 & 100m^4 \\ \hline
20m^2-9n^2 & -180m^2n^2+81n^4 \\
 & -180m^2n^2+81n^4 \\ \hline
\end{array}$$

14.
$$\begin{array}{r|l}
 & 64a^2b^2-112abx^2+49x^4 \quad \lfloor 8ab-7x^2 \\
 & 64a^2b^2 \\ \hline
16ab-7x^2 & -112abx^2+49x^4 \\
 & -112abx^2+49x^4 \\ \hline
\end{array}$$

15.
$$\begin{array}{r|l}
 & 121x^4y^2+110x^2yz^2+25z^4 \quad \lfloor 11x^2y+5z^2 \\
 & 121x^4y^2 \\ \hline
22x^2y+5z^2 & 110x^2yz^2+25z^4 \\
 & 110x^2yz^2+25z^4 \\ \hline
\end{array}$$

16.
$$\begin{array}{r|l}
 & 25a^2x^4-20ax^2y^3+4y^6 \quad \lfloor 5ax^2-2y^3 \\
 & 25a^2x^4 \\ \hline
10ax^2-2y^3 & -20ax^2y^3+4y^6 \\
 & -20ax^2y^3+4y^6 \\ \hline
\end{array}$$

17.
$$\begin{array}{r|l}
 & a^4-2a^3+3a^2-2a+1 \quad \lfloor a^2-a+1 \\
 & a^4 \\ \hline
2a^2-a & -2a^3+3a^2 \\
 & -2a^3+a^2 \\ \hline
2a^2-2a+1 & 2a^2-2a+1 \\
 & 2a^2-2a+1 \\ \hline
\end{array}$$

18.

$$\begin{array}{r|l|l}
 & 4y^4+12y^3+5y^2-6y+1 & 2y^2+3y-1 \\
 & 4y^4 & \\
4y^2+3y & 12y^3+5y^2 & \\
 & 12y^3+9y^2 & \\
4y^2+6y-1 & -4y^2-6y+1 & \\
 & -4y^2-6y+1 &
\end{array}$$

19.

$$\begin{array}{r|l|l}
 & 4x^4-4x^3+5x^2-2x+1 & 2x^2-x+1 \\
 & 4x^4 & \\
4x^2-x & -4x^3+5x^2 & \\
 & -4x^3+\ \ x^2 & \\
4x^2-2x+1 & 4x^2-2x+1 & \\
 & 4x^2-2x+1 &
\end{array}$$

20.

$$\begin{array}{r|l|l}
 & 4a^4-12a^3y+25a^2y^2-24ay^3+16y^4 & 2a^2-3ay+4y^2 \\
 & 4a^4 & \\
4a^2-3ay & -12a^3y+25a^2y^2 & \\
 & -12a^3y+\ \ 9a^2y^2 & \\
4a^2-6ay+4y^2 & 16a^2y^2-24ay^3+16y^4 & \\
 & 16a^2y^2-24ay^3+16y^4 &
\end{array}$$

21.

$$\begin{array}{r|l|l}
 & 25m^4-30m^3n+49m^2n^2-24mn^3+16n^4 & 5m^2-3mn+4n^2 \\
 & 25m^4 & \\
10m^2-3mn & -30m^3n+49m^2n^2 & \\
 & -30m^3n+\ \ 9m^2n^2 & \\
10m^2-6mn+4n^2 & 40m^2n^2-24mn^3+16n^4 & \\
 & 40m^2n^2-24mn^3+16n^4 &
\end{array}$$

22.

$$\begin{array}{r|l|l}
 & & a^3-3a^2b+3ab^2-b^3 \\
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6 & \\
 & a^6 & \\
3a^3-3a^2b & -6a^5b+15a^4b^2 & \\
 & -6a^5b+\ \ 9a^4b^2 & \\
2a^3-6a^2b+3ab^2 & 6a^4b^2-20a^3b^3+15a^2b^4 & \\
 & 6a^4b^2-18a^3b^3+\ \ 9a^2b^4 & \\
2a^3-6a^2b+6ab^2-b^3 & -\ \ 2a^3b^3+\ \ 6a^2b^4-6ab^5+b^6 & \\
 & -\ \ 2a^3b^3+\ \ 6a^2b^4-6ab^5+b^6 &
\end{array}$$

23.

$$x^6-6x^5+17x^4-34x^3+46x^2-40x+25 \quad \underline{|\, x^3-3x^2+4x-5}$$
$$x^6$$
$$\underline{2x^3-3x^2\,|} \quad -6x^5+17x^4$$
$$-6x^5+\ 9x^4$$
$$\underline{2x^3-6x^2+4x\,|} \quad 8x^4-34x^3+46x^2$$
$$8x^4-24x^3+16x^2$$
$$\underline{2x^3-6x^2+8x-5\,|} \quad -10x^3+30x^2-40x+25$$
$$-10x^3+30x^2-40x+25$$

$$|\, \underline{2x^3-3x^2y-5xy^2+7y^3}$$

24.

$$4x^6-12x^5y-11x^4y^2+58x^3y^3-17x^2y^4-70xy^5+49y^6$$
$$4x^6$$
$$\underline{4x^3-3x^2y\,|} \quad -12x^5y-11x^4y^2$$
$$-12x^5y+\ 9x^4y^2$$
$$\underline{4x^3-6x^2y-5xy^2\,|} \quad -20x^4y^2+58x^3y^3-17x^2y^4$$
$$-20x^4y^2+30x^3y^3+25x^2y^4$$
$$\underline{4x^3-6x^2y-10xy^2+7y^3\,|} \quad 28x^3y^3-42x^2y^4-70xy^5+49y^6$$
$$28x^3y^3-42x^2y^4-70xy^5+49y^6$$

$$|\, \underline{m^3-5m^2n-3mn^2-6n^3}$$

25.

$$m^6-10m^5n+19m^4n^2+18m^3n^3+69m^2n^4+36mn^5+36n^6$$
$$m^6$$
$$\underline{2m^3-5m^2n\,|} \quad -10m^5n+19m^4n^2$$
$$-10m^5n+25m^4n^2$$
$$\underline{2m^3-10m^2n-3mn^2\,|} \quad -\ 6m^4n^2+18m^3n^3+69m^2n^4$$
$$-\ 6m^4n^2+30m^3n^3+\ 9m^2n^4$$
$$\underline{2m^3-10m^2n-6mn^2-6n^3\,|} \quad -12m^3n^3+60m^2n^4+36mn^5+36n^6$$
$$-12m^3n^3+60m^2n^4+36mn^5+36n^6$$

Art. 251.

1.

$$x^3-3x^2y+3xy^2-y^3 \quad \underline{|\, x-y}$$
$$x^3$$
$$\underline{3x^2-3xy+y^2\,|} \quad -3x^2y+3xy^2-y^3$$
$$-3x^2y+3xy^2-y^3$$

2.
$$\begin{array}{r|l|l} & m^3+3m^2n+3mn^2+n^3 & m+n \\ & m^3 & \\ \hline 3m^2+3mn+n^2 & 3m^2n+3mn^2+n^3 & \\ & 3m^2n+3mn^2+n^3 & \end{array}$$

3.
$$\begin{array}{r|l|l} & x^3+9x^2+27x+27 & x+3 \\ & x^3 & \\ \hline 3x^2+9x+9 & 9x^2+27x+27 & \\ & 9x^2+27x+27 & \end{array}$$

4.
$$\begin{array}{r|l|l} & 8a^3+36a^2+54a+27 & 2a+3 \\ & 8a^3 & \\ \hline 12a^2+18a+9 & 36a^2+54a+27 & \\ & 36a^2+54a+27 & \end{array}$$

5.
$$\begin{array}{r|l|l} & 27x^3-54x^2+36x-8 & 3x-2 \\ & 27x^3 & \\ \hline 27x^2-18x+4 & -54x^2+36x-8 & \\ & -54x^2+36x-8 & \end{array}$$

6.
$$\begin{array}{r|l|l} & x^6-6x^4+12x^2-8 & x^2-2 \\ & x^6 & \\ \hline 3x^4-6x^2+4 & -6x^4+12x^2-8 & \\ & -6x^4+12x^2-8 & \end{array}$$

7.
$$\begin{array}{r|l|l} & 27y^6-135y^4+225y^2-125 & 3y^2-5 \\ & 27y^6 & \\ \hline 27y^4-45y^2+25 & -135y^4+225y^2-125 & \\ & -135y^4+225y^2-125 & \end{array}$$

8.
$$\begin{array}{r|l|l} & a^3x^3+6a^2x^2+12ax+8 & ax+2 \\ & a^3x^3 & \\ \hline 3a^2x^2+6ax+4 & 6a^2x^2+12ax+8 & \\ & 6a^2x^2+12ax+8 & \end{array}$$

9.
$$\begin{array}{r|l|l} & 125x^6-75x^4+15x^2-1 & 5x^2-1 \\ & 125x^6 & \\ \hline 75x^4-15x^2+1 & -75x^4+15x^2-1 & \\ & -75x^4+15x^2-1 & \end{array}$$

10.

$$8x^6-60x^4y^3+150x^2y^6-125y^9 \;\;\big|\; \underline{2x^2-5y^3}$$
$$8x^6$$
$$\underline{12x^4-30x^2y^3+25y^6} \;\big|\; -60x^4y^3+150x^2y^6-125y^9$$
$$\underline{-60x^4y^3+150x^2y^6-125y^9}$$

11.

$$125a^6b^3-225a^4b^2+135a^2b-27 \;\;\big|\; \underline{5a^2b-3}$$
$$125a^6b^3$$
$$\underline{75a^4b^2-45a^2b+9} \;\big|\; -225a^4b^2+135a^2b-27$$
$$\underline{-225a^4b^2-135a^2b-27}$$

12.

$$x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \;\;\big|\; \underline{x^2-2x+1}$$
$$x^6$$
$$3x^4-6x^3+4x^2 \;\big|\; -6x^5+15x^4-20x^3$$
$$-6x^5+12x^4-\;\;8x^3$$
$$3x^4-12x^3+12x^2 \;\big|\; 3x^4-12x^3+15x^2-6x+1$$
$$3x^2-6x+1$$
$$\underline{3x^4-12x^3+15x^2-6x+1} \;\big|\; \underline{3x^4-12x^3+15x^2-6x+1}$$

13.

$$x^6-9x^5+33x^4-63x^3+66x^2-36x+8 \;\;\big|\; \underline{x^2-3x+2}$$
$$x^6$$
$$3x^4-9x^3+9x^2 \;\big|\; -9x^5+33x^4-63x^3$$
$$-9x^5+27x^4-27x^3$$
$$3x^4-18x^3+27x^2 \;\big|\; 6x^4-36x^3+66x^2-36x+8$$
$$6x^2-18x+4$$
$$\underline{3x^4-18x^3+33x^2-18x+4} \;\big|\; \underline{6x^4-36x^4+66x^2-36x+8}$$

14.

$$\big|\; \underline{2x^2-3ax+4a^2}$$
$$8x^6-36ax^5+102a^2x^4-171a^3x^3+204a^4x^2-144a^5x+64a^6$$
$$8x^6$$
$$12x^4-18ax^3+9a^2x^2 \;\big|\; -36ax^5+102a^2x^4-171a^3x^3$$
$$-36ax^5+\;\;54a^2x^4-\;\;27a^3x^3$$
$$12x^4-36ax^3+27a^2x^2 \;\big|\; 48a^2x^4-144a^3x^3+204a^4x^2-144a^5x+64a^6$$
$$24a^2x^2-36a^3x+16a^4$$
$$\underline{12x^4-36ax^3+51a^2x^2-36a^3x+16a^4} \;\big|\; \underline{48a^2x^4-144a^3x^3+204a^4x^2-144a^5x+64a^6}$$

15.

$$\begin{array}{l|l}
 & \underline{2m^2+3mn-n^2} \\
 & 8m^6+36m^5n+42m^4n^2-9m^3n^3-21m^2n^4+9mn^5-n^6 \\
 & 8m^6 \\
\hline
12m^4+18m^3n+9m^2n^2 & 36m^5n+42m^4n^2-9m^3n^3 \\
 & 36m^5n+54m^4n^2+27m^3n^3 \\
\hline
12m^4+36m^3n+27m^2n^2 & -12m^4n^2-36m^3n^3-21m^2n^4+9mn^5-n^6 \\
\qquad -6m^2n^2-9mn^3+n^4 & \\
\hline
12m^4+36m^3n+21m^2n^2-9mn^3+n^4 & -12m^4n^2-36m^3n^3-21m^2n^4+9mn^5-n^6 \\
\hline
\end{array}$$

16.

$$\begin{array}{l|l}
 & \underline{3a^2-5a-2} \\
 & 27a^6-135a^5+171a^4+55a^3-114a^2-60a-8 \\
 & 27a^6 \\
\hline
27a^4-45a^3+25a^2 & -135a^5+171a^4+55a^3 \\
 & -135a^5+225a^4-125a^3 \\
\hline
27a^4-90a^3+75a^2 & -54a^4+180a^3-114a^2-60a-8 \\
\qquad -18a^2+30a+4 & \\
\hline
27a^4-90a^3+57a^2+30a+4 & -54a^4+180a^3-114a^2-60a-8 \\
\hline
\end{array}$$

17.

$$\begin{array}{l|l}
 & \underline{6a^4-4a^2-3} \\
 & 216a^{12}-432a^{10}-36a^8+368a^6+18a^4-108a^2-27 \\
 & 216a^{12} \\
\hline
108a^8-72a^6+16a^4 & -432a^{10}-36a^8+368a^6 \\
 & -432a^{10}+288a^8-64a^6 \\
\hline
108a^8-144a^6+48a^4 & -324a^8+432a^6+18a^4-108a^2-27 \\
\qquad -54a^4+36a^2+9 & \\
\hline
108a^8-144a^6-6a^4+36a^2+9 & -324a^8+432a^6+18a^4-108a^2-27 \\
\hline
\end{array}$$

18.

$$\begin{array}{l|l}
 & \underline{2x^2-3xy+5y^2} \\
 & 8x^6-36x^5y+114x^4y^2-207x^3y^3+285x^2y^4-225xy^5+125y^6 \\
 & 8x^6 \\
\hline
12x^4-18x^3y+9x^2y^2 & -36x^5y+114x^4y^2-207x^3y^3 \\
 & -36x^5y+54x^4y^2-27x^3y^3 \\
\hline
12x^4-36x^3y+27x^2y^2 & 60x^4y^2-180x^3y^3+285x^2y^4-225xy^5+125y^6 \\
\qquad 30x^2y^2-45xy^3+25y^4 & \\
\hline
12x^4-36x^3y+57x^2y^2-45xy^3+25y^4 & 60x^4y^2-180x^3y^3+285x^2y^4-225xy^5+125y^6 \\
\hline
\end{array}$$

19.

$$\begin{array}{l|l}
 & 27a^6-108a^5-45a^4+440a^3+105a^2-588a-343 \quad \underline{|\,3a^2-4a-7} \\
 & 27a^6 \\
27a^4-36a^3+16a^2 & -108a^5-45a^4+440a^3 \\
 & -108a^5+144a^4-64a^3 \\
\hline
27a^4-72a^3+48a^2 & -189a^4+504a^3+105a^2-588a-343 \\
\quad -63a^2+84a+49 & \\
\hline
27a^4-72a^3-15a^2+84a+49 & -189a^4+504a^3+105a^2-588a-343
\end{array}$$

20.

$$\begin{array}{l|l}
 & 8x^6+48ax^5+60a^2x^4-80a^3x^3-90a^4x^2+108a^5x-27a^6 \quad \underline{|\,2x^2+4ax-3a^2} \\
 & 8x^6 \\
12x^4+24ax^3+16a^2x^2 & 48ax^5+60a^2x^4-80a^3x^3 \\
 & 48ax^5+96a^2x^4+64a^3x^3 \\
\hline
12x^4+48ax^3+48a^2x^2 & -36a^2x^4-144a^3x^3-90a^4x^2+108a^5x-27a^6 \\
\quad -18a^2x^2-36a^3x+9a^4 & \\
\hline
12x^4+48ax^3+30a^2x^2-36a^3x+9a^4 & -36a^2x^4-144a^3x^3-90a^4x^2+108a^5x-27a^6
\end{array}$$

Art. 263.

10.

$$\begin{array}{l}
x^{-5}-17x^{-3}+7x^{-2}-9 \quad \underline{|\,x^{-2}+5x^{-1}+3} \\
\underline{x^{-5}+5x^{-4}+3x^{-3}} \quad x^{-3}-5x^{-2}+5x^{-1}-3 \\
\quad -5x^{-4}-20x^{-3}+7x^{-2}-9 \\
\quad \underline{-5x^{-4}-25x^{-3}-15x^{-2}} \\
\qquad 5x^{-3}+22x^{-2}-9 \\
\qquad \underline{5x^{-3}+25x^{-2}+15x^{-1}} \\
\qquad\quad -3x^{-2}-15x^{-1}-9 \\
\qquad\quad \underline{-3x^{-2}-15x^{-1}-9}
\end{array}$$

11.

$$
\begin{array}{l|l}
a^{-1}-82a^{-\frac{1}{5}}-3 & a^{-\frac{1}{5}}+3 \\
\cline{2-2}
\underline{a^{-1}+3a^{-\frac{4}{5}}} & a^{-\frac{4}{5}}-3a^{-\frac{3}{5}}+9a^{-\frac{2}{5}}-27a^{-\frac{1}{5}}-1 \\
\quad -3a^{-\frac{4}{5}}-82a^{-\frac{1}{5}}-3 & \\
\quad \underline{-3a^{-\frac{4}{5}}-\ 9a^{-\frac{3}{5}}} & \\
\qquad 9a^{-\frac{3}{5}}-82a^{-\frac{1}{5}}-3 & \\
\qquad \underline{9a^{-\frac{3}{5}}+27a^{-\frac{2}{5}}} & \\
\qquad\quad -27a^{-\frac{2}{5}}-82a^{-\frac{1}{5}}-3 & \\
\qquad\quad \underline{-27a^{-\frac{2}{5}}-81a^{-\frac{1}{5}}} & \\
\qquad\qquad -\ a^{-\frac{1}{5}}-3 & \\
\qquad\qquad \underline{-\ a^{-\frac{1}{5}}-3} &
\end{array}
$$

12.

$$
\begin{array}{l|l}
m^{2n}+m^nx^p+x^{2p} & m^n-m^{\frac{n}{2}}x^{\frac{p}{2}}+x^p \\
\cline{2-2}
\underline{m^{2n}-m^{\frac{3n}{2}}x^{\frac{p}{2}}+m^nx^p} & m^n+m^{\frac{n}{2}}x^{\frac{p}{2}}+x^p \\
\quad m^{\frac{3n}{2}}x^{\frac{p}{2}}+x^{2p} & \\
\quad \underline{m^{\frac{3n}{2}}x^{\frac{p}{2}}-m^nx^p+m^{\frac{n}{2}}x^{\frac{3p}{2}}} & \\
\qquad m^nx^p+m^{\frac{n}{2}}x^{\frac{3p}{2}}+x^{2p} & \\
\qquad \underline{m^nx^p+m^{\frac{n}{2}}x^{\frac{3p}{2}}+x^{2p}} &
\end{array}
$$

13.

$$
\begin{array}{l|l}
4x^3+11x^2+21x+6x^{\frac{1}{2}}-10 & 2x-3x^{\frac{1}{2}}+5 \\
\cline{2-2}
\underline{4x^3-6x^{\frac{5}{2}}+10x^2} & 2x^2+3x^{\frac{3}{2}}+5x-2 \\
\quad 6x^{\frac{5}{2}}+\ \ x^2+21x+6x^{\frac{1}{2}}-10 & \\
\quad \underline{6x^{\frac{5}{2}}-\ 9x^2+15x^{\frac{3}{2}}} & \\
\qquad 10x^2-15x^{\frac{3}{2}}+21x+6x^{\frac{1}{2}}-10 & \\
\qquad \underline{10x^2-15x^{\frac{3}{2}}+25x} & \\
\qquad\qquad -\ 4x+6x^{\frac{1}{2}}-10 & \\
\qquad\qquad \underline{-\ 4x+6x^{\frac{1}{2}}-10} &
\end{array}
$$

14.

$$\begin{array}{l|l}
2x^2-9x^{\frac{4}{3}}-8x-1 & x+3x^{\frac{2}{3}}+3x^{\frac{1}{3}}+1 \\ \cline{2-2}
2x^2+6x^{\frac{5}{3}}+6x^{\frac{4}{3}}+2x & 2x-6x^{\frac{2}{3}}+3x^{\frac{1}{3}}-1 \\ \hline
-6x^{\frac{5}{3}}-15x^{\frac{4}{3}}-10x-1 & \\
-6x^{\frac{5}{3}}-18x^{\frac{4}{3}}-18x-6x^{\frac{2}{3}} & \\ \hline
3x^{\frac{4}{3}}+\ 8x+6x^{\frac{2}{3}}-1 & \\
3x^{\frac{4}{3}}+\ 9x+9x^{\frac{2}{3}}+3x^{\frac{1}{3}} & \\ \hline
-\ x-3x^{\frac{2}{3}}-3x^{\frac{1}{3}}-1 & \\
-\ x-3x^{\frac{2}{3}}-3x^{\frac{1}{3}}-1 & \\ \hline
\end{array}$$

15.

$$\begin{array}{l|l}
x-17x^{\frac{2}{3}}+13x^{\frac{1}{3}}+15x^{\frac{1}{6}}-12 & x^{\frac{1}{3}}-5x^{\frac{1}{6}}+4 \\ \cline{2-2}
x-5x^{\frac{5}{6}}+4x^{\frac{2}{3}} & x^{\frac{2}{3}}+5x^{\frac{1}{2}}+4x^{\frac{1}{3}}-3 \\ \hline
5x^{\frac{5}{6}}-21x^{\frac{2}{3}}+13x^{\frac{1}{3}}+15x^{\frac{1}{6}}-12 & \\
5x^{\frac{5}{6}}-25x^{\frac{2}{3}}+20x^{\frac{1}{2}} & \\ \hline
4x^{\frac{2}{3}}-20x^{\frac{1}{2}}+13x^{\frac{1}{3}}+15x^{\frac{1}{6}}-12 & \\
4x^{\frac{2}{3}}-20x^{\frac{1}{2}}+16x^{\frac{1}{3}} & \\ \hline
-\ 3x^{\frac{1}{3}}+15x^{\frac{1}{6}}-12 & \\
-\ 3x^{\frac{1}{3}}+15x^{\frac{1}{6}}-12 & \\ \hline
\end{array}$$

16.

$$\begin{array}{l|l}
x^{\frac{7}{3}}-109x-12x^{\frac{2}{3}}-56 & x-4x^{\frac{2}{3}}+3x^{\frac{1}{3}}-2 \\ \cline{2-2}
x^{\frac{7}{3}}-4x^2+3x^{\frac{5}{3}}-2x^{\frac{4}{3}} & x^{\frac{4}{3}}+4x+13x^{\frac{2}{3}}+42x^{\frac{1}{3}}+28 \\ \hline
4x^2-\ 3x^{\frac{5}{3}}+\ 2x^{\frac{4}{3}}-109x-12x^{\frac{2}{3}}-56 & \\
4x^2-16x^{\frac{5}{3}}+12x^{\frac{4}{3}}-\ \ 8x & \\ \hline
13x^{\frac{5}{3}}-10x^{\frac{4}{3}}-101x-\ 12x^{\frac{2}{3}}-56 & \\
13x^{\frac{5}{3}}-52x^{\frac{4}{3}}+\ 39x-\ 26x^{\frac{2}{3}} & \\ \hline
42x^{\frac{4}{3}}-140x+\ 14x^{\frac{2}{3}}-56 & \\
42x^{\frac{4}{3}}-168x+126x^{\frac{2}{3}}-84x^{\frac{1}{3}} & \\ \hline
28x-112x^{\frac{2}{3}}+84x^{\frac{1}{3}}-56 & \\
28x-112x^{\frac{2}{3}}+84x^{\frac{1}{3}}-56 & \\ \hline
\end{array}$$

17.

$$\begin{array}{l|l}
2a^{-1}-10a^{-\frac{3}{5}}-38a^{-\frac{2}{5}}-98 & a^{-\frac{2}{5}}+4a^{-\frac{1}{5}}+7 \\ \cline{2-2}
\underline{2a^{-1}+8a^{-\frac{4}{5}}+14a^{-\frac{3}{5}}} & 2a^{-\frac{3}{5}}-8a^{-\frac{2}{5}}+8a^{-\frac{1}{5}}-14 \\
\quad -8a^{-\frac{4}{5}}-24a^{-\frac{3}{5}}-38a^{-\frac{2}{5}}-98 & \\
\quad \underline{-8a^{-\frac{4}{5}}-32a^{-\frac{3}{5}}-56a^{-\frac{2}{5}}} & \\
\qquad 8a^{-\frac{3}{5}}+18a^{-\frac{2}{5}}-98 & \\
\qquad \underline{8a^{-\frac{3}{5}}+32a^{-\frac{2}{5}}+56a^{-\frac{1}{5}}} & \\
\qquad\quad -14a^{-\frac{2}{5}}-56a^{-\frac{1}{5}}-98 & \\
\qquad\quad \underline{-14a^{-\frac{2}{5}}-56a^{-\frac{1}{5}}-98} &
\end{array}$$

18.

$$\begin{array}{l|l}
8x^{-2}-95x^{-1}+1 & 2x^{-\frac{2}{3}}-5x^{-\frac{1}{3}}+1 \\ \cline{2-2}
\underline{8x^{-2}-20x^{-\frac{5}{3}}+\ 4x^{-\frac{4}{3}}} & 4x^{-\frac{4}{3}}+10x^{-1}+23x^{-\frac{2}{3}}+5x^{-\frac{1}{3}}+1 \\
\quad 20x^{-\frac{5}{3}}-\ 4x^{-\frac{4}{3}}-\ 95x^{-1}+1 & \\
\quad \underline{20x^{-\frac{5}{3}}-50x^{-\frac{4}{3}}+\ 10x^{-1}} & \\
\qquad 46x^{-\frac{4}{3}}-105x^{-1}+1 & \\
\qquad \underline{46x^{-\frac{4}{3}}-115x^{-1}+23x^{-\frac{2}{3}}} & \\
\qquad\quad 10x^{-1}-23x^{-\frac{2}{3}}+1 & \\
\qquad\quad \underline{10x^{-1}-25x^{-\frac{2}{3}}+5x^{-\frac{1}{3}}} & \\
\qquad\qquad 2x^{-\frac{2}{3}}-5x^{-\frac{1}{3}}+1 & \\
\qquad\qquad \underline{2x^{-\frac{2}{3}}-5x^{-\frac{1}{3}}+1} &
\end{array}$$

19.

$$\begin{array}{l|l}
x^{2}+x^{\frac{5}{3}}-11x^{\frac{1}{3}}+9 & x^{\frac{2}{3}}-2x^{\frac{1}{3}}+1 \\ \cline{2-2}
\underline{x^{2}-2x^{\frac{5}{3}}+x^{\frac{4}{3}}} & x^{\frac{4}{3}}+3x+5x^{\frac{2}{3}}+7x^{\frac{1}{3}}+9 \\
\quad 3x^{\frac{5}{3}}-\ x^{\frac{4}{3}}-11x^{\frac{1}{3}}+9 & \\
\quad \underline{3x^{\frac{5}{3}}-6x^{\frac{4}{3}}+\ 3x} & \\
\qquad 5x^{\frac{4}{3}}-\ 3x-11x^{\frac{1}{3}}+9 & \\
\qquad \underline{5x^{\frac{4}{3}}-10x+\ 5x^{\frac{2}{3}}} & \\
\qquad\quad 7x-\ 5x^{\frac{2}{3}}-11x^{\frac{1}{3}}+9 & \\
\qquad\quad \underline{7x-14x^{\frac{2}{3}}+\ 7x^{\frac{1}{3}}} & \\
\qquad\qquad 9x^{\frac{2}{3}}-18x^{\frac{1}{3}}+9 & \\
\qquad\qquad \underline{9x^{\frac{2}{3}}-18x^{\frac{1}{3}}+9} &
\end{array}$$

20.
$$\begin{array}{l|l}
x^{\frac{7}{4}}+7x^{\frac{1}{4}}-21x^{-\frac{3}{4}}-21x^{-\frac{5}{4}}-6x^{-\frac{7}{4}} & x^{\frac{3}{4}}+3x^{\frac{1}{4}}+3x^{-\frac{1}{4}}+x^{-\frac{3}{4}} \\
\underline{x^{\frac{7}{4}}+3x^{\frac{5}{4}}+3x^{\frac{3}{4}}+x^{\frac{1}{4}}} & \overline{x-3x^{\frac{1}{2}}+6-3x^{-\frac{1}{2}}-6x^{-1}} \\
\quad -3x^{\frac{5}{4}}-3x^{\frac{3}{4}}+6x^{\frac{1}{4}}-21x^{-\frac{3}{4}}-21x^{-\frac{5}{4}}-6x^{-\frac{7}{4}} & \\
\quad \underline{-3x^{\frac{5}{4}}-9x^{\frac{3}{4}}-9x^{\frac{1}{4}}-3x^{-\frac{1}{4}}} & \\
\qquad 6x^{\frac{3}{4}}+15x^{\frac{1}{4}}+3x^{-\frac{1}{4}}-21x^{-\frac{3}{4}}-21x^{-\frac{5}{4}}-6x^{-\frac{7}{4}} & \\
\qquad \underline{6x^{\frac{3}{4}}+18x^{\frac{1}{4}}+18x^{-\frac{1}{4}}+6x^{-\frac{3}{4}}} & \\
\qquad\quad -3x^{\frac{1}{4}}-15x^{-\frac{1}{4}}-27x^{-\frac{3}{4}}-21x^{-\frac{5}{4}}-6x^{-\frac{7}{4}} & \\
\qquad\quad \underline{-3x^{\frac{1}{4}}-9x^{-\frac{1}{4}}-9x^{-\frac{3}{4}}-3x^{-\frac{5}{4}}} & \\
\qquad\qquad -6x^{-\frac{1}{4}}-18x^{-\frac{3}{4}}-18x^{-\frac{5}{4}}-6x^{-\frac{7}{4}} & \\
\qquad\qquad \underline{-6x^{-\frac{1}{4}}-18x^{-\frac{3}{4}}-18x^{-\frac{5}{4}}-6x^{-\frac{7}{4}}} &
\end{array}$$

Art. 271.

1. $\dfrac{a^{-2}}{x^{-1}} \times \dfrac{a^2x}{a^2x} = \dfrac{x}{a^2}.$

2. $\dfrac{m^{-2}n^{-3}}{a^{-1}x^{-2}} \times \dfrac{am^2n^3x^2}{am^2n^3x^2} = \dfrac{ax^2}{m^2n^3}.$

3. $\dfrac{5x^{-4}y^{-n}}{7a^{-3}b^{-2}} \times \dfrac{a^3b^2x^4y^n}{a^3b^2x^4y^n} = \dfrac{5a^3b^2}{7x^4y^n}.$

4. $\dfrac{a^{-1}}{xy^{-1}} \times \dfrac{ay}{ay} = \dfrac{y}{ax}.$

5. $\dfrac{am^{-1}}{a+b^{-1}} \times \dfrac{mb}{mb} = \dfrac{ab}{amb+m}.$

6. $\dfrac{3a^3b^{-1}}{3^{-1}a^{-2}b^{-2}} \times \dfrac{3a^2b^2}{3a^2b^2} = 9a^5b.$

7. $\dfrac{3a^{-2}b^{-3}x}{5^{-1}x^{-2}y^{-1}} \times \dfrac{5a^2b^3x^2y}{5a^2b^3x^2y} = \dfrac{15x^3y}{a^2b^3}.$

8. $\dfrac{(a+1)^{-2}}{(a+1)^{-1}} \times \dfrac{(a+1)^2}{(a+1)^2} = \dfrac{1}{a+1}.$

9. $\dfrac{1}{a^{-1}+b^{-2}} \times \dfrac{ab^2}{ab^2} = \dfrac{ab^2}{b^2+a}.$

10. $\dfrac{a-2}{(a-2)^{-1}} \times \dfrac{a-2}{a-2} = a^2-4a+4.$

11. $\dfrac{x^{-2}+y^{-2}}{y^{-1}-x^{-1}} \times \dfrac{x^2y^2}{x^2y^2} = \dfrac{y^2+x^2}{x^2y-xy^2} = \dfrac{x^2+y^2}{xy\,(x-y)}.$

12. $\dfrac{a^{-1}+b^{-2}}{a^{-2}+b^{-1}} \times \dfrac{a^2b^2}{a^2b^2} = \dfrac{ab^2+a^2}{b^2+a^2b} = \dfrac{a\,(a+b^2)}{b\,(a^2+b)}.$

13. $\dfrac{x^{-3}-1}{x^{-1}-1} \times \dfrac{x^3}{x^3} = \dfrac{1-x^3}{x^2-x^3} = \dfrac{(1-x)\,(1+x+x^2)}{x^2\,(1-x)} = \dfrac{1+x+x^2}{x^2}.$

14. $\dfrac{a+b}{a^{-1}+b^{-1}} \times \dfrac{ab}{ab} = \dfrac{ab\,(a+b)}{b+a} = ab.$

15. $\dfrac{a^{-1}+b}{b^{-1}+a} \times \dfrac{ab}{ab} = \dfrac{b+ab^2}{a+a^2b} = \dfrac{b\,(ab+1)}{a\,(ab+1)} = \dfrac{b}{a}.$

16. $\dfrac{x-y}{y^{-1}-x^{-1}} \times \dfrac{xy}{xy} = \dfrac{xy(x-y)}{x-y} = xy.$

17. $\dfrac{a^{-m}-b^{-m}}{a^{-2m}-b^{-2m}} \times \dfrac{a^{2m}b^{2m}}{a^{2m}b^{2m}} = \dfrac{a^m b^{2m} - a^{2m}b^m}{b^{2m}-a^{2m}} = \dfrac{a^m b^m (b^m - a^m)}{(b^m-a^m)(b^m+a^m)} = \dfrac{a^m b^m}{a^m+b^m}$

18. $\dfrac{x^{-2}-y^{-2}}{x^{-4}-y^{-4}} \times \dfrac{x^4y^4}{x^4y^4} = \dfrac{x^2y^4-x^4y^2}{y^4-x^4} = \dfrac{x^2y^2(y^2-x^2)}{(y^2-x^2)(y^2+x^2)} = \dfrac{x^2y^2}{x^2+y^2}.$

19. $\dfrac{5a(x-1)^{-1}-3(x-1)^{-1}}{(x-1)^{-2}} = \dfrac{(5a-3)(x-1)^{-1}}{(x-1)^{-2}} \times \dfrac{(x-1)^2}{(x-1)^2}$

$= (5a-3)(x-1) = 5ax-5a-3x+3.$

20. $\dfrac{x^{-2}(x^{-1}-y^{-1})^{-1}-y^{-2}(x^{-1}-y^{-1})^{-1}}{x^{-1}(x^{-1}-y^{-1})^{-2}+y^{-1}(x^{-1}-y^{-1})^{-2}}$

$= \dfrac{(x^{-2}-y^{-2})(x^{-1}-y^{-1})^{-1}}{(x^{-1}+y^{-1})(x^{-1}-y^{-1})^{-2}} = \dfrac{(x^{-1}-y^{-1})}{(x^{-1}-y^{-1})^{-1}}$

Multiplying by $\dfrac{(x^{-1}-y^{-1})}{(x^{-1}-y^{-1})} = \left(\dfrac{x^{-1}-y^{-1}}{1}\right)^2 = x^{-2}-2x^{-1}y^{-1}+y^{-2}.$

Multiplying by $\dfrac{x^2y^2}{x^2y^2} = \dfrac{y^2-2xy+x^2}{x^2y^2}$, or $\dfrac{(y-x)^2}{x^2y^2}$, or $\dfrac{(x-y)^2}{x^2y^2}.$

Art. 281.

1. $\sqrt{12} = \sqrt{3\times 4} = 2\sqrt{3}.$
2. $\sqrt{20} = \sqrt{4\times 5} = 2\sqrt{5}.$
3. $\sqrt{18} = \sqrt{2\times 9} = 3\sqrt{2}.$
4. $\sqrt{27} = \sqrt{3\times 9} = 3\sqrt{3}.$
5. $\sqrt{32} = \sqrt{2\times 16} = 4\sqrt{2}.$
6. $\sqrt{48} = \sqrt{3\times 16} = 4\sqrt{3}.$
7. $\sqrt{50} = \sqrt{2\times 25} = 5\sqrt{2}.$
8. $\sqrt{45} = \sqrt{5\times 9} = 3\sqrt{5}.$
9. $\sqrt{63a^3} = \sqrt{9a^2\times 7a} = 3a\sqrt{7a}.$
10. $\sqrt{54a^2b} = \sqrt{6b\times 9a^2} = 3a\sqrt{6b}.$
11. $\sqrt{72a^4b^3} = \sqrt{36a^4b^2\times 2b} = 6a^2b\sqrt{2b}.$
12. $\sqrt{75x^2y^3} = \sqrt{25x^2y^2\times 3y} = 5xy\sqrt{3y}.$
13. $\sqrt{80m^4n^5} = \sqrt{16m^4n^4\times 5n} = 4m^2n^2\sqrt{5n}.$
14. $\sqrt{99a^3b} = \sqrt{9a^2\times 11ab} = 3a\sqrt{11ab}.$
15. $\sqrt{112m^2n^5} = \sqrt{16m^2n^4\times 7n} = 4mn^2\sqrt{7n}.$
16. $\sqrt{108a^3x^2} = \sqrt{36a^2x^2\times 3a} = 6ax\sqrt{3a}.$

17. $\sqrt{125x^6y^3} = \sqrt{25x^6y^2 \times 5y} = 5x^3y\sqrt{5y}.$

18. $\sqrt{98m^3n^4} = \sqrt{49m^2n^4 \times 2m} = 7mn^2\sqrt{2m}.$

19. $\sqrt{147p^5q^7} = \sqrt{49p^4q^6 \times 3pq} = 7p^2q^3\sqrt{3pq}.$

20. $\sqrt{192x^7y^5z} = \sqrt{64x^6y^4 \times 3xyz} = 8x^3y^2\sqrt{3xyz}.$

21. $\sqrt{405x^2y^3z} = \sqrt{81x^2y^2 \times 5yz} = 9xy\sqrt{5yz}.$

22. $\sqrt{500m^5n^3} = \sqrt{100m^4n^2 \times 5mn} = 10m^2n\sqrt{5mn}.$

23. $\sqrt[3]{54a^4} = \sqrt[3]{27a^3 \times 2a} = 3a\sqrt[3]{2a}.$

24. $\sqrt[3]{32x^3y^4} = \sqrt{8x^3y^3 \times 4y} = 2xy\sqrt[3]{4y}.$

25. $\sqrt[3]{40m^4n^2} = \sqrt[3]{8m^3 \times 5mn^2} = 2m\sqrt[3]{5mn^2}.$

26. $\sqrt[3]{192a^6b^3c} = \sqrt[3]{64a^6b^3 \times 3c} = 4a^2b\sqrt[3]{3c}.$

27. $\sqrt[3]{250m^4n^4} = \sqrt[3]{125m^3n^3 \times 2mn} = 5mn\sqrt[3]{2mn}.$

28. $\sqrt[3]{432x^5y^4} = \sqrt[3]{216x^3y^3 \times 2x^2y} = 6xy\sqrt[3]{2x^2y}.$

29. $\sqrt[4]{32a^8b^5c^3} = \sqrt[4]{16a^8b^4 \times 2bc^3} = 2a^2b\sqrt[4]{2bc^3}.$

30. $\sqrt[4]{162a^5x^4y} = \sqrt[4]{81a^4x^4 \times 2ay} = 3ax\sqrt[4]{2ay}.$

31. $\sqrt[4]{1250ab^5} = \sqrt[4]{625b^4 \times 2ab} = 5b\sqrt[4]{2ab}.$

32. $\sqrt[5]{64a^6x^7y} = \sqrt{32a^5x^5 \times 2ax^2y} = 2ax\sqrt[5]{2ax^2y}.$

33. $\sqrt[6]{1458a^6x^{12}} = \sqrt[6]{729a^6x^{12} \times 2} = 3ax^2\sqrt[6]{2}.$

34. $\sqrt{3a^2+6ab+3b^2} = \sqrt{3(a^2+2ab+b^2)} = (a+b)\sqrt{3}.$

35. $\sqrt{12a^2-36ab+27b^2} = \sqrt{3(4a^2-12ab+9b^2)} = (2a-3b)\sqrt{3}.$

36. $\sqrt{2ax^2-4axy+2ay^2} = \sqrt{2a(x^2-2xy+y^2)} = (x-y)\sqrt{2a}.$

37. $\sqrt{125a^2b-200ab+80b} = \sqrt{5b(25a^2-40b+16)} = (5a-4)\sqrt{5b}.$

38. $\sqrt{27ax^2-36ax+12a} = \sqrt{3a(9x^2-12ax+4)} = (3x-2)\sqrt{3a}.$

39. $\sqrt{12x^2+60xy+75y^2} = \sqrt{3(4x^2+20xy+25y^2)} = (2x+5y)\sqrt{3}.$

40. $\sqrt{18a^2-84ab+98b^2} = \sqrt{2(9a^2-42ab+49b^2)} = (3a-7b)\sqrt{2}.$

41. $\sqrt{45x^3-210x^2y+245xy^2} = \sqrt{5x(9x^2-42xy+49y^2)} = (3x-7y)\sqrt{5x}.$

42. $\sqrt{5x+20x^{\frac{1}{2}}y^{\frac{1}{3}}+20y^{\frac{2}{3}}} = \sqrt{5\,(x+4x^{\frac{1}{2}}y^{\frac{1}{3}}+4y^{\frac{2}{3}})} = (x^{\frac{1}{2}}+2y^{\frac{1}{3}})\sqrt{5}.$

43. $\sqrt{8a^4-64a^2+128} = \sqrt{2\,(4a^4-32a^26+4)} = (2a^2-8)\sqrt{2}.$

44. $\sqrt[3]{4x^3-36x^2+108x-108} = \sqrt[3]{4\,(x^3-9x^2+27x-27)} = (x-3)\sqrt[3]{4}.$

45. $\sqrt[3]{3x^3-9x^2y+9xy^2-3y^3} = \sqrt[3]{3\,(x^3-3x^2y+3xy^2-y^3)} = (x-y)\sqrt[3]{3}.$

Art. 282.

1. $\sqrt[4]{9} = \sqrt[4]{3^2} = 3^{\frac{2}{4}} = 3^{\frac{1}{2}} = \sqrt{3}.$

2. $\sqrt[4]{25} = \sqrt[4]{5^2} = 5^{\frac{2}{4}} = 5^{\frac{1}{2}} = \sqrt{5}.$

3. $\sqrt[4]{64} = \sqrt[4]{2^4\times 2^2} = 2^{\frac{4}{4}}\times 2^{\frac{2}{4}} = 2\times 2^{\frac{1}{2}} = 2\sqrt{2}.$

4. $\sqrt[4]{100} = \sqrt[4]{10^2} = 10^{\frac{2}{4}} = 10^{\frac{1}{2}} = \sqrt{10}.$

5. $\sqrt[6]{100} = \sqrt[6]{10^2} = 10^{\frac{2}{6}} = 10^{\frac{1}{3}} = \sqrt[3]{10}.$

6. $\sqrt[6]{64} = \sqrt[6]{8^2} = 8^{\frac{2}{6}} = 8^{\frac{1}{3}} = \sqrt[3]{8} = 2.$

7. $\sqrt[6]{27} = \sqrt[6]{3^3} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3}.$

8. $\sqrt[10]{32} = \sqrt[10]{2^5} = 2^{\frac{5}{10}} = 2^{\frac{1}{2}} = \sqrt{2}.$

9. $\sqrt[12]{64} = \sqrt[12]{2^6} = 2^{\frac{6}{12}} = 2^{\frac{1}{2}} = \sqrt{2}.$

10. $\sqrt[14]{128} = \sqrt[14]{2^7} = 2^{\frac{7}{14}} = 2^{\frac{1}{2}} = \sqrt{2}.$

11. $\sqrt[9]{343} = \sqrt[9]{7^3} = 7^{\frac{3}{9}} = 7^{\frac{1}{3}} = \sqrt[3]{7}.$

12. $\sqrt[6]{216} = \sqrt[6]{6^3} = 6^{\frac{3}{6}} = 6^{\frac{1}{2}} = \sqrt{6}.$

13. $\sqrt[12]{256} = \sqrt[12]{2^8} = 2^{\frac{8}{12}} = 2^{\frac{2}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4}.$

14. $\sqrt[15]{243} = \sqrt[15]{3^5} = 3^{\frac{5}{15}} = 3^{\frac{1}{3}} = \sqrt[3]{3}.$

15. $\sqrt[6]{125} = \sqrt[6]{5^3} = 5^{\frac{3}{6}} = 5^{\frac{1}{2}} = \sqrt{5}.$

16. $\sqrt[9]{216} = \sqrt[9]{6^3} = 6^{\frac{3}{9}} = 6^{\frac{1}{3}} = \sqrt[3]{6}.$

17. $\sqrt[12]{81a^4} = \sqrt[12]{3^4a^4} = (3a)^{\frac{4}{12}} = 3a^{\frac{1}{3}} = \sqrt[3]{3a}.$

18. $\sqrt[15]{32a^5b^{10}} = \sqrt[15]{(2ab^2)^5} = (2ab^2)^{\frac{5}{15}} = (2ab^2)^{\frac{1}{3}} = \sqrt[3]{2ab^2}.$

19. $\sqrt[10]{243x^5y^{15}} = \sqrt[10]{(3xy^3)^5} = (3xy^3)^{\frac{5}{10}} = (3xy^3)^{\frac{1}{2}} = y\sqrt{3xy}.$

20. $\sqrt[9]{1000a^3b^6} = \sqrt[9]{(10ab^2)^3} = (10ab^2)^{\frac{3}{9}} = (10ab^2)^{\frac{1}{3}} = \sqrt[3]{10ab^2}$.

21. $\sqrt[6]{1000x^3y^9} = \sqrt[6]{(10xy)^3 \times y^6} = (10xy)^{\frac{3}{6}} \times y^{\frac{6}{6}} = (10xy)^{\frac{1}{2}} \times y = y\sqrt{10xy}$.

22. $\sqrt[12]{1000a^6b^3} = \sqrt[12]{(10a^2b)^3} = (10a^2b)^{\frac{3}{12}} = (10a^2b)^{\frac{1}{4}} = \sqrt[4]{10a^2b}$.

23. $\sqrt[18]{512a^9b^{18}} = \sqrt[18]{(2ab^2)^9} = (2ab^2)^{\frac{9}{18}} = (2ab^2)^{\frac{1}{2}} = b\sqrt{2a}$.

24. $\sqrt[12]{625x^4y^8} = \sqrt[12]{(5xy^2)^4} = (5xy^2)^{\frac{4}{12}} = (5xy^2)^{\frac{1}{3}} = \sqrt[3]{5xy^2}$.

25. $\sqrt[6]{5^3 \times 3^3a^6b^9} = \sqrt[6]{(5 \times 3 \times b)^3 \times (ab)^6} = (5 \times 3 \times b)^{\frac{3}{6}} \times (ab)^{\frac{6}{6}} = ab\sqrt{15b}$.

26. $\sqrt[9]{3^3 \times 5^3 \times 2^3} = \sqrt[9]{(3 \times 5 \times 2)^3} = (3 \times 5 \times 2)^{\frac{3}{9}} = 30^{\frac{1}{3}} = \sqrt[3]{30}$.

27. $\sqrt[8]{5^3 \times 5a^4x^8} = \sqrt[8]{(5a)^4 \times x^8} = (5a)^{\frac{4}{8}} \times x^{\frac{8}{8}} = (5a)^{\frac{1}{2}} \times x = x\sqrt{5a}$.

28. $\sqrt[10]{3^5 \times 2^5a^5m^{10}} = \sqrt[10]{(6a)^5 \times m^{10}} = (6a)^{\frac{5}{10}} \times m^{\frac{10}{10}} = (6a)^{\frac{1}{2}} \times m = m\sqrt{6a}$.

29. $\sqrt[12]{19^2 \times 19^4a^6b^{12}} = \sqrt[12]{(19a)^6 \times b^{12}} = (19a)^{\frac{6}{12}} \times b^{\frac{12}{12}} = (19a)^{\frac{1}{2}} \times b = b\sqrt{19a}$.

30. $\sqrt[10]{10^2 \times 10^3a^5x^{10}} = \sqrt[10]{(10a)^5 \times x^{10}} = (10a)^{\frac{5}{10}} \times x^{\frac{10}{10}} = (10a)^{\frac{1}{2}} \times x$
$= x\sqrt{10a}$.

31. $\sqrt[8]{11^4 \times 7^4a^4b^8c^{12}} = \sqrt[8]{(77ac)^4 \times b^8c^8} = (77ac)^{\frac{4}{8}} \times (bc)^{\frac{8}{8}} = (77ac)^{\frac{1}{2}} \times bc$
$= bc\sqrt{77ac}$.

32. $\sqrt[6]{5^3 \times 7^3 \times 3^3a^6b^3} = \sqrt[6]{(105b)^3 \times a^6} = (105b)^{\frac{3}{6}} \times a^{\frac{6}{6}} = (105b)^{\frac{1}{2}} \times a$
$= a\sqrt{105b}$.

Art. 283.

1. $\sqrt{\frac{3}{8}} = \sqrt{\frac{3}{8} \times \frac{2}{2}} = \sqrt{\frac{6}{16}} = \sqrt{6} \times \sqrt{\frac{1}{16}} = \frac{1}{4}\sqrt{6}$.

2. $\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3} \times \frac{3}{3}} = \sqrt{\frac{6}{9}} = \sqrt{6} \times \sqrt{\frac{1}{9}} = \frac{1}{3}\sqrt{6}$.

3. $\sqrt{\frac{5}{6}} = \sqrt{\frac{5}{6} \times \frac{6}{6}} = \sqrt{\frac{30}{36}} = \sqrt{30} \times \sqrt{\frac{1}{36}} = \frac{1}{6}\sqrt{30}$.

4. $\sqrt{\frac{7}{12}} = \sqrt{\frac{7}{12} \times \frac{3}{3}} = \sqrt{\frac{21}{36}} = \sqrt{21} \times \sqrt{\frac{1}{36}} = \frac{1}{6}\sqrt{21}$.

5. $\sqrt{\frac{11}{18}} = \sqrt{\frac{11}{18} \times \frac{2}{2}} = \sqrt{\frac{22}{36}} = \sqrt{22} \times \sqrt{\frac{1}{36}} = \frac{1}{6}\sqrt{22}$.

6. $\sqrt{\frac{19}{50}} = \sqrt{\frac{19}{50} \times \frac{2}{2}} = \sqrt{\frac{38}{100}} = \sqrt{38} \times \sqrt{\frac{1}{100}} = \frac{1}{10}\sqrt{38}$.

7. $\sqrt{\frac{3}{7}} = \sqrt{\frac{3}{7} \times \frac{7}{7}} = \sqrt{\frac{21}{49}} = \sqrt{21} \times \sqrt{\frac{1}{49}} = \frac{1}{7}\sqrt{21}$.

8. $\sqrt[3]{\frac{1}{2}} = \sqrt[3]{\frac{1}{2} \times \frac{4}{4}} = \sqrt[3]{4} \times \sqrt[3]{\frac{1}{8}} = \frac{1}{2}\sqrt[3]{4}$.

9. $\sqrt[3]{\frac{7}{9}} = \sqrt[3]{\frac{7}{9} \times \frac{3}{3}} = \sqrt{21} \times \sqrt{\frac{1}{27}} = \frac{1}{3}\sqrt[3]{21}.$

10. $\sqrt[3]{\frac{11}{16}} = \sqrt[3]{\frac{11}{16} \times \frac{4}{4}} = \sqrt[3]{44} \times \sqrt[3]{\frac{1}{64}} = \frac{1}{4}\sqrt[3]{44}.$

11. $\sqrt[3]{\frac{14}{25}} = \sqrt[3]{\frac{14}{25} \times \frac{5}{5}} = \sqrt[3]{70} \times \sqrt[3]{\frac{1}{125}} = \frac{1}{5}\sqrt[3]{70}.$

12. $\sqrt[3]{\frac{35}{72}} = \sqrt[3]{\frac{35}{72} \times \frac{3}{3}} = \sqrt[3]{105} \times \sqrt[3]{216} = \frac{1}{6}\sqrt[3]{105}.$

13. $\sqrt[3]{\frac{16}{49}} = \sqrt[3]{\frac{16}{49} \times \frac{7}{7}} = \sqrt[3]{14} \times \sqrt[3]{\frac{8}{343}} = \frac{2}{7}\sqrt[3]{14}.$

14. $\sqrt[3]{\frac{5}{9}} = \sqrt[3]{\frac{5}{9} \times \frac{3}{3}} = \sqrt[3]{15} \times \sqrt[3]{\frac{1}{27}} = \frac{1}{3}\sqrt[3]{15}.$

15. $\sqrt[4]{\frac{7}{8}} = \sqrt[4]{\frac{7}{8} \times \frac{2}{2}} = \sqrt[4]{14} \times \sqrt[4]{\frac{1}{16}} = \frac{1}{2}\sqrt[4]{14}.$

16. $\sqrt[4]{\frac{19}{27}} = \sqrt[4]{\frac{19}{27} \times \frac{3}{3}} = \sqrt[4]{57} \times \sqrt[4]{\frac{1}{81}} = \frac{1}{3}\sqrt[4]{57}.$

17. $2\sqrt{\frac{1}{2}} = 2\sqrt{\frac{1}{2} \times \frac{2}{2}} = \sqrt{2} \times 2\sqrt{\frac{1}{4}} = \sqrt{2}.$

18. $3\sqrt{\frac{2}{3}} = 3\sqrt{\frac{2}{3} \times \frac{3}{3}} = \sqrt{6} \times 3\sqrt{\frac{1}{9}} = \sqrt{6}.$

19. $5\sqrt{\frac{4}{5}} = 5\sqrt{\frac{4}{5} \times \frac{5}{5}} = 5\sqrt{5} \times \sqrt{\frac{4}{25}} = 2\sqrt{5}.$

20. $2\sqrt[3]{\frac{1}{2}} = 2\sqrt[3]{\frac{1}{2} \times \frac{4}{4}} = 2\sqrt[3]{4} \times \sqrt[3]{\frac{1}{8}} = \sqrt[3]{4}.$

21. $\frac{2}{3}\sqrt[3]{\frac{3}{4}} = \frac{2}{3}\sqrt[3]{\frac{3}{4} \times \frac{2}{2}} = \frac{2}{3}\sqrt[3]{6} \times \sqrt[3]{\frac{1}{8}} = \frac{1}{3}\sqrt[3]{6}.$

22. $\frac{5}{6}\sqrt[3]{\frac{4}{5}} = \frac{5}{6}\sqrt[3]{\frac{4}{5} \times \frac{25}{25}} = \frac{5}{6}\sqrt[3]{100} \times \sqrt[3]{\frac{1}{125}} = \frac{1}{6}\sqrt[3]{100}.$

23. $\frac{4b}{3a}\sqrt[3]{\frac{5a^4}{16b^2}} = \frac{4b}{3a}\sqrt[3]{\frac{5a^4}{16b^2} \times \frac{4b}{4b}} = \frac{4b}{3a}\sqrt[3]{20ab} \times \sqrt[3]{\frac{a^3}{64b^3}} = \frac{1}{3}\sqrt[3]{20ab}.$

24. $\frac{5}{4}\sqrt[3]{\frac{8x^6}{25a^2}} = \frac{5}{4}\sqrt[3]{\frac{8x^6}{25a^2} \times \frac{5a}{5a}} = \frac{5}{4}\sqrt[3]{5a} \times \sqrt[3]{\frac{8x^6}{125a^3}} = \frac{x^2}{2a}\sqrt[3]{5a}.$

25. $\frac{3}{2}\sqrt{\frac{4a^2b}{5xy^2}} = \frac{3}{2}\sqrt{\frac{4a^2b}{5xy^2} \times \frac{5x}{5x}} = \frac{3}{2}\sqrt{5bx} \times \sqrt{\frac{4a^2}{25x^2y^2}} = \frac{3a}{5xy}\sqrt{5bx}.$

26. $\frac{10}{3}\sqrt{\frac{9mn^2}{10x^2y^3}} = \frac{10}{3}\sqrt{\frac{9mn^2}{10x^2y^3} \times \frac{10y}{10y}} = \frac{10}{3}\sqrt{10my} \times \sqrt{\frac{9n^2}{100x^2y^4}}$

$= \frac{n}{xy^2}\sqrt{10my}.$

27. $5\frac{1}{2}\sqrt{\frac{10ab^2}{11x^2y}} = \frac{11}{2}\sqrt{\frac{10ab^2}{11x^2y} \times \frac{11y}{11y}} = \frac{11}{2}\sqrt{110ay} \times \sqrt{\frac{b^2}{121x^2y^2}}$

$= \frac{b}{2xy}\sqrt{110ay}.$

28. $8\frac{2}{3}\sqrt{\frac{9a^2b}{13x^3y^5}} = \frac{26}{3}\sqrt{\frac{9a^2b}{13x^3y^5} \times \frac{13xy}{13xy}} = \frac{26}{3}\sqrt{13bxy} \times \sqrt{\frac{9a^2}{169x^4y^6}}$

$= \frac{2a}{x^2y^3}\sqrt{13bxy}.$

29. $\frac{3a^3b}{x^2y}\sqrt[3]{\frac{5x^3y^4}{6a^4b^3}} = \frac{3a^3b}{x^2y}\sqrt[3]{\frac{5x^3y^4}{6a^4b^3} \times \frac{36a^2}{36a^2}} = \frac{3a^3b}{x^2y}\sqrt{180a^2y} \times \sqrt[3]{\frac{x^3y^3}{216a^6b^3}}$

$= \frac{a}{2x}\sqrt[3]{180a^2y}.$

30. $\frac{2a}{3b}\sqrt{4\frac{1}{2}b^4} = \frac{2a}{3b}\sqrt{\frac{9b^4}{2} \times \frac{2}{2}} = \frac{2a}{3b}\sqrt{2} \times \sqrt{\frac{9b^4}{4}} = ab\sqrt{2}.$

31. $2\frac{1}{4}\sqrt{5\frac{1}{3}a^4} = \frac{9}{4}\sqrt{\frac{16a^4}{3} \times \frac{3}{3}} = \frac{9}{4}\sqrt{3} \times \sqrt{\frac{16a^4}{9}} = 3a^2\sqrt{3}.$

32. $1\frac{2}{5}\sqrt{\frac{25a^3}{28b}} = \frac{7}{5}\sqrt{\frac{25a^3}{28b} \times \frac{7b}{7b}} = \frac{7}{5}\sqrt{7ab} \times \sqrt{\frac{25a^2}{196b^2}} = \frac{a}{2b}\sqrt{7ab}.$

33. $(x-y)\sqrt{\frac{x+y}{x-y}} = (x-y)\sqrt{\frac{x+y}{x-y} \times \frac{x-y}{x-y}}$

$= (x-y)\sqrt{x^2-y^2} \times \sqrt{\frac{1}{(x-y)^2}} = \sqrt{x^2-y^2}.$

34. $\frac{a+x}{b}\sqrt{\frac{ab^2}{4(a+x)}} = \frac{a+x}{b}\sqrt{\frac{ab^2}{4(a+x)} \times \frac{a+x}{a+x}}$

$= \frac{a+x}{b}\sqrt{a(a+x)} \times \sqrt{\frac{b^2}{4(a+x)^2}} = \frac{1}{2}\sqrt{a(a+x)}.$

35. $\frac{a^2-x^2}{x}\sqrt{\frac{ax^2}{a-x}} = \frac{a^2-x^2}{x}\sqrt{\frac{ax^2}{a-x} \times \frac{a-x}{a-x}}$

$= \frac{a^2-x^2}{x}\sqrt{a(a-x)} \times \sqrt{\frac{x^2}{(a-x)^2}} = (a+x)\sqrt{a(a-x)}.$

36. $5(x-y)\sqrt{\frac{x^2+2xy+y^2}{5(x^2-y^2)}} = 5(x-y)\sqrt{\frac{x+y}{5(x-y)} \times \frac{5(x-y)}{5(x-y)}}$

$= 5(x-y)\sqrt{5(x^2-y^2)} \times \sqrt{\frac{1}{[5(x-y)]^2}} = \sqrt{5(x^2-y^2)}.$

Art. 285.

1. $5\sqrt{\tfrac{19}{25}} = \sqrt{(5)^2} \times \sqrt{\tfrac{19}{25}} = \sqrt{19}.$

2. $3\sqrt{5} = \sqrt{(3)^2} \times \sqrt{5} = \sqrt{45}.$

3. $\tfrac{2}{3}\sqrt{18} = \sqrt{(\tfrac{2}{3})^2} \times \sqrt{18} = \sqrt{8}.$

4. $2\tfrac{2}{3}\sqrt{45} = \sqrt{(\tfrac{8}{3})^2} \times \sqrt{45} = \sqrt{320}.$

5. $3\tfrac{3}{4}\sqrt{\tfrac{32}{45}} = \sqrt{(\tfrac{15}{4})^2} \times \sqrt{\tfrac{32}{45}} = \sqrt{10}.$

6. $3\sqrt[3]{2} = \sqrt[3]{(3)^3} \times \sqrt[3]{2} = \sqrt[3]{54}.$

7. $5\sqrt[3]{\tfrac{19}{25}} = \sqrt[3]{(5)^3} \times \sqrt[3]{\tfrac{19}{25}} = \sqrt[3]{95}.$

8. $2\sqrt[4]{2} = \sqrt[4]{(2)^4} \times \sqrt[4]{2} = \sqrt[4]{32}.$

9. $3\sqrt[3]{2ab} = \sqrt[3]{(3)^3} \times \sqrt[3]{2ab} = \sqrt[3]{54ab}.$

10. $2a^2b\sqrt[3]{ab^2} = \sqrt[3]{(2a^2b)^3} \times \sqrt[3]{ab^2} = \sqrt[3]{8a^7b^5}.$

11. $3ab^2\sqrt{\dfrac{2}{ab}} = \sqrt{(3ab^2)^2} \times \sqrt{\dfrac{2}{ab}} = \sqrt{18ab^3}.$

12. $2\tfrac{1}{2}a\sqrt[3]{\dfrac{64x}{25a^2}} = \sqrt[3]{\left(\dfrac{5a}{2}\right)^3} \times \sqrt[3]{\dfrac{64x}{25a^2}} = \sqrt[3]{40ax}.$

13. $2\tfrac{2}{3}mn^2\sqrt{5\tfrac{5}{8}} = \sqrt{\left(\dfrac{8mn^2}{3}\right)^2} \times \sqrt{\dfrac{45}{8}} = \sqrt{40m^2n^4}.$

14. $\dfrac{2}{3a}\sqrt[4]{7\tfrac{7}{8}a^5} = \sqrt[4]{\left(\dfrac{2}{3a}\right)^4} \times \sqrt[4]{\dfrac{63a^5}{8}} = \sqrt[4]{\dfrac{14a}{9}}.$

15. $3x^2y\sqrt[4]{\dfrac{1}{27x^3y^3}} = \sqrt[4]{(3x^2y)^4} \times \sqrt[4]{\dfrac{1}{27x^3y^3}} = \sqrt[4]{3x^5y}.$

16. $(x-3)\sqrt{\dfrac{x+3}{x-3}} = \sqrt{(x-3)^2} \times \sqrt{\dfrac{x+3}{x-3}} = \sqrt{x^2-9}.$

17. $\dfrac{a+b}{a-b}\sqrt{\dfrac{a-b}{a+b}} = \sqrt{\left(\dfrac{a+b}{a-b}\right)^2} \times \sqrt{\dfrac{a-b}{a+b}} = \sqrt{\dfrac{a+b}{a-b}}.$

18. $(x+3)\sqrt{1-\frac{6}{x+3}}=\sqrt{(x+3)^2}\times\sqrt{\frac{x-3}{x+3}}=\sqrt{x^2-9}.$

19. $\frac{x+a}{x-a}\sqrt{1-\frac{2a}{x+a}}=\sqrt{\left(\frac{x+a}{x-a}\right)^2}\times\sqrt{\frac{x-a}{x+a}}=\sqrt{\frac{x+a}{x-a}}.$

Art. 287.

1. $\left.\begin{matrix}\sqrt{3}\\ \sqrt[3]{4}\end{matrix}\right\}=\left\{\begin{matrix}3^{\frac{1}{2}}\\ 4^{\frac{1}{3}}\end{matrix}\right\}=\left\{\begin{matrix}3^{\frac{3}{6}}\\ 4^{\frac{2}{6}}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[6]{3^3}\\ \sqrt[6]{4^2}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[6]{27},\\ \sqrt[6]{16}.\end{matrix}\right.$

2. $\left.\begin{matrix}\sqrt{5}\\ \sqrt[3]{7}\end{matrix}\right\}=\left\{\begin{matrix}5^{\frac{1}{2}}\\ 7^{\frac{1}{3}}\end{matrix}\right\}=\left\{\begin{matrix}5^{\frac{3}{6}}\\ 7^{\frac{2}{6}}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[6]{5^3}\\ \sqrt[6]{7^2}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[6]{125},\\ \sqrt[6]{49}.\end{matrix}\right.$

3. $\left.\begin{matrix}\sqrt[3]{2}\\ \sqrt[4]{3}\end{matrix}\right\}=\left\{\begin{matrix}2^{\frac{1}{3}}\\ 3^{\frac{1}{4}}\end{matrix}\right\}=\left\{\begin{matrix}2^{\frac{4}{12}}\\ 3^{\frac{3}{12}}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[12]{2^4}\\ \sqrt[12]{3^3}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[12]{16},\\ \sqrt[12]{27}.\end{matrix}\right.$

4. $\left.\begin{matrix}\sqrt{5}\\ \sqrt[4]{10}\end{matrix}\right\}=\left\{\begin{matrix}5^{\frac{1}{2}}\\ 10^{\frac{1}{4}}\end{matrix}\right\}=\left\{\begin{matrix}5^{\frac{2}{4}}\\ 10^{\frac{1}{4}}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[4]{5^2}\\ \sqrt[4]{10}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[4]{25},\\ \sqrt[4]{10}.\end{matrix}\right.$

5. $\left.\begin{matrix}\sqrt{2}\\ \sqrt[3]{3}\\ \sqrt[6]{7}\end{matrix}\right\}=\left\{\begin{matrix}2^{\frac{1}{2}}\\ 3^{\frac{1}{3}}\\ 7^{\frac{1}{6}}\end{matrix}\right\}=\left\{\begin{matrix}2^{\frac{3}{6}}\\ 3^{\frac{2}{6}}\\ 7^{\frac{1}{6}}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[6]{2^3}\\ \sqrt[6]{3^2}\\ \sqrt[6]{7}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[6]{8},\\ \sqrt[6]{9},\\ \sqrt[6]{7}.\end{matrix}\right.$

6. $\left.\begin{matrix}\sqrt{2}\\ \sqrt[3]{4}\\ \sqrt[4]{10}\end{matrix}\right\}=\left\{\begin{matrix}2^{\frac{1}{2}}\\ 4^{\frac{1}{3}}\\ 10^{\frac{1}{4}}\end{matrix}\right\}=\left\{\begin{matrix}2^{\frac{6}{12}}\\ 4^{\frac{4}{12}}\\ 10^{\frac{3}{12}}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[12]{2^6}\\ \sqrt[12]{4^4}\\ \sqrt[12]{10^3}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[12]{64},\\ \sqrt[12]{256},\\ \sqrt[12]{1000}.\end{matrix}\right.$

7. $\left.\begin{matrix}\sqrt[10]{2}\\ \sqrt{3}\\ \sqrt[5]{4}\end{matrix}\right\}=\left\{\begin{matrix}2^{\frac{1}{10}}\\ 3^{\frac{1}{2}}\\ 4^{\frac{1}{5}}\end{matrix}\right\}=\left\{\begin{matrix}2^{\frac{1}{10}}\\ 3^{\frac{5}{10}}\\ 4^{\frac{2}{10}}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[10]{2}\\ \sqrt[10]{3^5}\\ \sqrt[10]{4^2}\end{matrix}\right\}=\left\{\begin{matrix}\sqrt[10]{2},\\ \sqrt[10]{243},\\ \sqrt[10]{16}.\end{matrix}\right.$

8. $$\left.\begin{array}{r}\sqrt[3]{2}\\ \sqrt[6]{5}\\ \sqrt[9]{11}\end{array}\right\}=\left\{\begin{array}{c}2^{\frac{1}{3}}\\ 5^{\frac{1}{6}}\\ 11^{\frac{1}{9}}\end{array}\right\}=\left\{\begin{array}{c}2^{\frac{6}{18}}\\ 5^{\frac{3}{18}}\\ 11^{\frac{2}{18}}\end{array}\right\}=\left\{\begin{array}{c}\sqrt[18]{2^6}\\ \sqrt[18]{5^3}\\ \sqrt[18]{11^2}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[18]{64},\\ \sqrt[18]{125},\\ \sqrt[18]{121}.\end{array}\right.$$

9. $$\left.\begin{array}{r}\sqrt{2}\\ \sqrt[3]{4}\\ \sqrt[4]{5}\\ \sqrt[6]{7}\end{array}\right\}=\left\{\begin{array}{c}2^{\frac{1}{2}}\\ 4^{\frac{1}{3}}\\ 5^{\frac{1}{4}}\\ 7^{\frac{1}{6}}\end{array}\right\}=\left\{\begin{array}{c}2^{\frac{6}{12}}\\ 4^{\frac{4}{12}}\\ 5^{\frac{3}{12}}\\ 7^{\frac{2}{12}}\end{array}\right\}=\left\{\begin{array}{c}\sqrt[12]{2^6}\\ \sqrt[12]{4^4}\\ \sqrt[12]{5^3}\\ \sqrt[12]{7^2}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[12]{64},\\ \sqrt[12]{256},\\ \sqrt[12]{125},\\ \sqrt[12]{49}.\end{array}\right.$$

10. $$\left.\begin{array}{r}\sqrt{2}\\ \sqrt[5]{5}\\ \sqrt[4]{3}\\ \sqrt[10]{12}\end{array}\right\}=\left\{\begin{array}{c}2^{\frac{1}{2}}\\ 5^{\frac{1}{5}}\\ 3^{\frac{1}{4}}\\ 12^{\frac{1}{10}}\end{array}\right\}=\left\{\begin{array}{c}2^{\frac{10}{20}}\\ 5^{\frac{4}{20}}\\ 3^{\frac{5}{20}}\\ 12^{\frac{2}{20}}\end{array}\right\}=\left\{\begin{array}{c}\sqrt[20]{2^{10}}\\ \sqrt[20]{5^4}\\ \sqrt[20]{3^5}\\ \sqrt[20]{12^2}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[20]{1024},\\ \sqrt[20]{625},\\ \sqrt[20]{243},\\ \sqrt[20]{144}.\end{array}\right.$$

11. $$\left.\begin{array}{r}\sqrt[3]{a}\\ \sqrt[4]{b}\\ \sqrt[6]{c}\\ \sqrt[8]{ax}\end{array}\right\}=\left\{\begin{array}{r}a^{\frac{1}{3}}\\ b^{\frac{1}{4}}\\ c^{\frac{1}{6}}\\ (ax)^{\frac{1}{8}}\end{array}\right\}=\left\{\begin{array}{r}a^{\frac{8}{24}}\\ b^{\frac{6}{24}}\\ c^{\frac{4}{24}}\\ (ax)^{\frac{3}{24}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[24]{a^8},\\ \sqrt[24]{b^6},\\ \sqrt[24]{c^4},\\ \sqrt[24]{a^3x^3}.\end{array}\right.$$

12. $$\left.\begin{array}{r}\sqrt{xy}\\ \sqrt[3]{xy^2}\\ \sqrt[4]{a^2b}\\ \sqrt[6]{ab^2}\end{array}\right\}=\left\{\begin{array}{c}(xy)^{\frac{1}{2}}\\ (xy^2)^{\frac{1}{3}}\\ (a^2b)^{\frac{1}{4}}\\ (ab^2)^{\frac{1}{6}}\end{array}\right\}=\left\{\begin{array}{c}(xy)^{\frac{6}{12}}\\ (xy^2)^{\frac{4}{12}}\\ (a^2b)^{\frac{3}{12}}\\ (ab^2)^{\frac{2}{12}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[12]{x^6y^6},\\ \sqrt[12]{x^4y^8},\\ \sqrt[12]{a^6b^3},\\ \sqrt[12]{a^2b^4}.\end{array}\right.$$

13. $$\left.\begin{array}{r}\sqrt{a+x}\\ \sqrt[3]{x-y}\end{array}\right\}=\left\{\begin{array}{c}(a+x)^{\frac{1}{2}}\\ (x-y)^{\frac{1}{3}}\end{array}\right\}=\left\{\begin{array}{c}(a+x)^{\frac{3}{6}}\\ (x-y)^{\frac{2}{6}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[6]{(a+x)^3},\\ \sqrt[6]{(x-y)^2}.\end{array}\right.$$

14. $$\left.\begin{array}{r}\sqrt[4]{m-n}\\ \sqrt[6]{m+n}\end{array}\right\}=\left\{\begin{array}{c}(m-n)^{\frac{1}{4}}\\ (m+n)^{\frac{1}{6}}\end{array}\right\}=\left\{\begin{array}{c}(m-n)^{\frac{3}{12}}\\ (m+n)^{\frac{2}{12}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[12]{(m-n)^3},\\ \sqrt[12]{(m+n)^2}.\end{array}\right.$$

15. $$\left.\begin{array}{l}\sqrt[6]{x+y}\\ \sqrt[8]{m+1}\end{array}\right\}=\left\{\begin{array}{l}(x+y)^{\frac{1}{6}}\\ (m+1)^{\frac{1}{8}}\end{array}\right\}=\left\{\begin{array}{l}(x+y)^{\frac{4}{24}}\\ (m+1)^{\frac{3}{24}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[24]{(x+y)^4},\\ \sqrt[24]{(m+1)^3}.\end{array}\right.$$

16. $$\left.\begin{array}{l}\sqrt[9]{x^2-1}\\ \sqrt[6]{x^2+1}\end{array}\right\}=\left\{\begin{array}{l}(x^2-1)^{\frac{1}{9}}\\ (x^2+1)^{\frac{1}{6}}\end{array}\right\}=\left\{\begin{array}{l}(x^2-1)^{\frac{2}{18}}\\ (x^2+1)^{\frac{3}{18}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[18]{(x^2-1)^2},\\ \sqrt[18]{(x^2+1)^3}.\end{array}\right.$$

17. $$\left.\begin{array}{l}\sqrt{18}\\ \sqrt[3]{16}\\ \sqrt[4]{162}\\ \sqrt[6]{128}\end{array}\right\}=\left\{\begin{array}{r}18^{\frac{1}{2}}\\ 16^{\frac{1}{3}}\\ 162^{\frac{1}{4}}\\ 128^{\frac{1}{6}}\end{array}\right\}=\left\{\begin{array}{r}18^{\frac{6}{12}}\\ 16^{\frac{4}{12}}\\ 162^{\frac{3}{12}}\\ 128^{\frac{2}{12}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[12]{18^6}\\ \sqrt[12]{16^4}\\ \sqrt[12]{162^3}\\ \sqrt[12]{128^2}\end{array}\right\}=\left\{\begin{array}{l}3\sqrt[12]{64},\\ 2\sqrt[12]{16},\\ 3\sqrt[12]{8},\\ 2\sqrt[12]{4}.\end{array}\right.$$

18. $$\left.\begin{array}{l}\sqrt{75}\\ \sqrt[3]{54}\\ \sqrt[4]{405}\\ \sqrt[6]{1458}\end{array}\right\}=\left\{\begin{array}{r}75^{\frac{1}{2}}\\ 54^{\frac{1}{3}}\\ 405^{\frac{1}{4}}\\ 1458^{\frac{1}{6}}\end{array}\right\}=\left\{\begin{array}{r}75^{\frac{6}{12}}\\ 54^{\frac{4}{12}}\\ 405^{\frac{3}{12}}\\ 1458^{\frac{2}{12}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[12]{75^6}\\ \sqrt[12]{54^4}\\ \sqrt[12]{405^3}\\ \sqrt[12]{1458^2}\end{array}\right\}=\left\{\begin{array}{l}5\sqrt[12]{729},\\ 3\sqrt[12]{16},\\ 3\sqrt[12]{125},\\ 3\sqrt[12]{4}.\end{array}\right.$$

19. $$\left.\begin{array}{l}\sqrt{200}\\ \sqrt[3]{375}\\ \sqrt[6]{5103}\\ \sqrt[9]{5632}\end{array}\right\}=\left\{\begin{array}{r}200^{\frac{1}{2}}\\ 375^{\frac{1}{3}}\\ 5103^{\frac{1}{6}}\\ 5632^{\frac{1}{9}}\end{array}\right\}=\left\{\begin{array}{r}200^{\frac{9}{18}}\\ 375^{\frac{6}{18}}\\ 5103^{\frac{3}{18}}\\ 5632^{\frac{2}{18}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[18]{200^9}\\ \sqrt[18]{375^6}\\ \sqrt[18]{5103^3}\\ \sqrt[18]{5632^2}\end{array}\right\}=\left\{\begin{array}{r}10\sqrt[18]{512},\\ 5\sqrt[18]{729},\\ 3\sqrt[18]{343},\\ 2\sqrt[18]{121}.\end{array}\right.$$

Art. 289.

1. $$\left.\begin{array}{l}+\sqrt{242}\\ +3\sqrt{18}\end{array}\right\}=\left\{\begin{array}{l}+11\sqrt{2}\\ +\ \ 9\sqrt{2}\end{array}\right\}=(11+9)\sqrt{2}=20\sqrt{2}.$$

2. $$\left.\begin{array}{l}+\sqrt{147}\\ -\sqrt{75}\end{array}\right\}=\left\{\begin{array}{l}+7\sqrt{3}\\ -5\sqrt{3}\end{array}\right\}=(7-5)\sqrt{3}=2\sqrt{3}.$$

3. $$\left.\begin{array}{l}+\sqrt{108}\\ +\sqrt{432}\end{array}\right\}=\left\{\begin{array}{l}+\ \ 6\sqrt{3}\\ +12\sqrt{3}\end{array}\right\}=(6+12)\sqrt{3}=18\sqrt{3}.$$

4. $\left.\begin{array}{l} +2\sqrt{96} \\ +5\sqrt{216} \end{array}\right\} = \left\{\begin{array}{l} +\ 8\sqrt{6} \\ +30\sqrt{6} \end{array}\right\} = (8+30)\sqrt{6} = 38\sqrt{6}.$

5. $\left.\begin{array}{l} +5\sqrt[3]{320} \\ -3\sqrt[3]{135} \end{array}\right\} = \left\{\begin{array}{l} +20\sqrt[3]{5} \\ -\ 9\sqrt[3]{5} \end{array}\right\} = (20-9)\sqrt[3]{5} = 11\sqrt[3]{5}.$

6. $\left.\begin{array}{l} +\sqrt[3]{432} \\ +\sqrt[3]{250} \end{array}\right\} = \left\{\begin{array}{l} +6\sqrt[3]{2} \\ +5\sqrt[3]{2} \end{array}\right\} = (6+5)\sqrt[3]{2} = 11\sqrt[3]{2}.$

7. $\left.\begin{array}{l} +3\sqrt{\frac{2}{5}} \\ -2\sqrt{\frac{1}{10}} \end{array}\right\} = \left\{\begin{array}{l} +6\sqrt{\frac{1}{10}} \\ -2\sqrt{\frac{1}{10}} \end{array}\right\} = \left\{\begin{array}{l} +\frac{3}{5}\sqrt{10} \\ -\frac{1}{5}\sqrt{10} \end{array}\right\} = (\tfrac{3}{5}-\tfrac{1}{5})\sqrt{10} = \tfrac{2}{5}\sqrt{10}.$

8. $\left.\begin{array}{l} +\sqrt[3]{500} \\ +\sqrt[3]{108} \end{array}\right\} = \left\{\begin{array}{l} +5\sqrt[3]{4} \\ +3\sqrt[3]{4} \end{array}\right\} = (5+3)\sqrt[3]{4} = 8\sqrt[3]{4}.$

9. $\left.\begin{array}{l} +\sqrt{605} \\ -\sqrt{405} \end{array}\right\} = \left\{\begin{array}{l} +11\sqrt{5} \\ -\ 9\sqrt{5} \end{array}\right\} = (11-9)\sqrt{5} = 2\sqrt{5}.$

10. $\left.\begin{array}{l} +\ 4\sqrt[3]{\frac{3}{4}} \\ +10\sqrt[3]{\frac{48}{125}} \end{array}\right\} = \left\{\begin{array}{l} +\ 4\sqrt[3]{\frac{48}{64}} \\ +10\sqrt[3]{\frac{48}{125}} \end{array}\right\} = \left\{\begin{array}{l} +2\sqrt[3]{6} \\ +4\sqrt[3]{6} \end{array}\right\} = (4+2)\sqrt[3]{6} = 6\sqrt[3]{6}.$

11. $\left.\begin{array}{l} +\ \ \sqrt{125} \\ +5\sqrt{405} \\ -\ \ \sqrt{500} \end{array}\right\} = \left\{\begin{array}{l} +\ 5\sqrt{5} \\ +45\sqrt{5} \\ -10\sqrt{5} \end{array}\right\} = (5+45-10)\sqrt{5} = 40\sqrt{5}.$

12. $\left.\begin{array}{l} +\ 6\sqrt{\frac{3}{4}} \\ +10\sqrt{\frac{12}{25}} \\ -14\sqrt{\frac{27}{49}} \end{array}\right\} = \left\{\begin{array}{l} +\ 6\sqrt{\frac{12}{16}} \\ +10\sqrt{\frac{12}{25}} \\ -14\sqrt{\frac{27}{49}} \end{array}\right\} = \left\{\begin{array}{l} +3\sqrt{3} \\ +4\sqrt{3} \\ -6\sqrt{3} \end{array}\right\} = (3+4-6)\sqrt{3} = +\sqrt{3}.$

13. $\left.\begin{array}{l} +20\sqrt{\frac{4}{5}} \\ -\ \ \sqrt{245} \\ +35\sqrt{1\frac{4}{5}} \end{array}\right\} = \left\{\begin{array}{l} +\ 8\sqrt{5} \\ -\ 7\sqrt{5} \\ +21\sqrt{5} \end{array}\right\} = (8-7+21)\sqrt{5} = 22\sqrt{5}.$

14. $$\left.\begin{array}{l} +\ 8\sqrt{\frac{3}{4}} \\ +\ \ \sqrt{60} \\ -2\frac{1}{5}\sqrt{15} \\ +\ \ \sqrt{\frac{3}{5}} \end{array}\right\} = \left\{\begin{array}{l} +\ 4\sqrt{3} \\ +\ 2\sqrt{15} \\ -\frac{11}{5}\sqrt{15} \\ +\ \frac{1}{5}\sqrt{15} \end{array}\right\} = (\tfrac{10}{5}-\tfrac{11}{5}+\tfrac{1}{5})\sqrt{15}+4\sqrt{3} = 4\sqrt{3}.$$

15. $$\left.\begin{array}{l} +\sqrt{243xy^2} \\ +\sqrt{192xy^2} \\ -\sqrt{363xy^2} \end{array}\right\} = \left\{\begin{array}{l} +\ 9y\sqrt{3x} \\ +\ 8y\sqrt{3x} \\ -11y\sqrt{3x} \end{array}\right\} = (9y+8y-11y)\sqrt{3x} = 6y\sqrt{3x}.$$

16. $$\left.\begin{array}{l} +\sqrt{18a^5} \\ +\sqrt{50a^3} \\ -\sqrt{162a^3} \end{array}\right\} = \left\{\begin{array}{l} +3a^2\sqrt{2a} \\ +\ 5a\sqrt{2a} \\ -\ 9a\sqrt{2a} \end{array}\right\} = (3a^2+5a-9a)\sqrt{2a} = (3a^2-4a)\sqrt{2a}.$$

17. $$\left.\begin{array}{l} +\sqrt{2ax^2+4ax+2a} \\ -\sqrt{2ax^2-4ax+2a} \end{array}\right\} = \left\{\begin{array}{l} +(x+1)\sqrt{2a} \\ -(x-1)\sqrt{2a} \end{array}\right\} = (x+1)-(x-1)\sqrt{2a}$$
$$= 2\sqrt{2a}.$$

18. $$\left.\begin{array}{l} +\ \sqrt[3]{81} \\ -2\sqrt[3]{24} \\ +\ \sqrt{28} \\ +2\sqrt{63} \end{array}\right\} = \left\{\begin{array}{l} +3\sqrt[3]{3} \\ -4\sqrt[3]{3} \\ +2\sqrt{7} \\ +6\sqrt{7} \end{array}\right\} = (3-4)\sqrt[3]{3}+(2+6)\sqrt{7} = 8\sqrt{7}-\sqrt[3]{3}.$$

19. $$\left.\begin{array}{l} +\sqrt[3]{a^7x} \\ +\sqrt[3]{8a^4x^4} \\ +\sqrt[3]{ax^7} \end{array}\right\} = \left\{\begin{array}{l} +\ a^2\sqrt[3]{ax} \\ +2ax\sqrt[3]{ax} \\ +\ x^2\sqrt[3]{ax} \end{array}\right\} = (a^2+2ax+x^2)\sqrt[3]{ax} = (a+x)^2\sqrt[3]{ax}.$$

20. $$\left.\begin{array}{l} +\ \sqrt[3]{x^3y^4-x^3y^3z} \\ +x\sqrt[3]{y^4-y^3z} \end{array}\right\} = \left\{\begin{array}{l} +xy\sqrt[3]{y-z} \\ +xy\sqrt[3]{y-z} \end{array}\right\} = 2xy\sqrt[3]{y-z}.$$

Art. 291.

1. $\sqrt{5} \times \sqrt{50} = \sqrt{5 \times 2 \times 25} = 5\sqrt{10}.$

2. $2\sqrt{3} \times 3\sqrt{12} = 6\sqrt{36} = 6 \times 6 = 36.$

3. $4\sqrt{6} \times 3\sqrt{8} = 12\sqrt{48} = 12\sqrt{3 \times 16} = 48\sqrt{3}.$

4. $2\sqrt[3]{4} \times \sqrt[3]{16} = 2\sqrt[3]{64} = 2 \times 4 = 8.$

5. $5\sqrt[3]{6} \times \sqrt[3]{18} = 5\sqrt[3]{108} = 5\sqrt[3]{4 \times 27} = 15\sqrt[3]{4}.$

6. $4\sqrt[4]{8} \times \sqrt[4]{12} = 4\sqrt[4]{96} = 4\sqrt[4]{6 \times 16} = 8\sqrt[4]{6}.$

7. $5\sqrt[4]{27} \times \sqrt[4]{45} = 5\sqrt[4]{27 \times 3 \times 15} = 5\sqrt[4]{81 \times 15} = 15\sqrt[4]{15}.$

8. $ab\sqrt{ab^2} \times \sqrt{abx} = ab\sqrt{a^2b^2 \times bx} = a^2b^2\sqrt{bx}.$

9. $2x\sqrt[3]{x^2y} \times 3y\sqrt[3]{xy^2} = 6xy\sqrt[3]{x^3y^3} = 6x^2y^2.$

10. $5x\sqrt{18a^2b} \times 3\sqrt[3]{3ab^2x} = 15x\sqrt[3]{27a^3b^3 \times 2x} = 45abx\sqrt[3]{2x}.$

11. $7ab\sqrt[4]{a^3b^2} \times 2\sqrt[4]{a^2b^3} = 14ab\sqrt[4]{a^4b^4 \times ab} = 14a^2b^2\sqrt[4]{ab}.$

12. $2\sqrt[4]{36x^3y^5} \times 3\sqrt[4]{72x^5y^3} = 6\sqrt[4]{4 \times 9 \times 8 \times 9x^8y^8} = 6\sqrt[4]{16 \times 81x^8y^8 \times 2}$
$= 36x^2y^2\sqrt[4]{2}.$

13. $3\sqrt[3]{36ab} \times 4\sqrt[3]{24a^2b} = 12\sqrt[3]{6 \times 6 \times 6a^3 \times 4b^2} = 72a\sqrt[3]{4b^2}.$

14. $5\sqrt[3]{75} \times 3\sqrt[3]{45} = 15\sqrt[3]{5 \times 5 \times 5 \times 3^3} = 225.$

15.

$$\begin{array}{r} \sqrt{2}+\sqrt{3} \\ \sqrt{2}+\sqrt{3} \\ \hline 2+\sqrt{6} \\ \sqrt{6}+3 \\ \hline 2+2\sqrt{6}+3 = \\ 5+2\sqrt{6}. \end{array}$$

16.

$$\begin{array}{r} \sqrt{5}-\sqrt{3} \\ \sqrt{5}-\sqrt{3} \\ \hline 5-\sqrt{15} \\ -\sqrt{15}+3 \\ \hline 5-2\sqrt{15}+3 = \\ 8-2\sqrt{15}. \end{array}$$

17.

$$\begin{array}{r}\sqrt{10}+\sqrt{6}\\ \sqrt{10}-\sqrt{6}\\ \hline 10+\sqrt{60}\qquad\\ -\sqrt{60}-6\\ \hline 10\qquad -6=4.\end{array}$$

18.

$$\begin{array}{r}\sqrt{5}-2\sqrt{3}\\ 2\sqrt{5}-3\sqrt{3}\\ \hline 10-4\sqrt{15}\qquad\\ -3\sqrt{15}+18\\ \hline 10-7\sqrt{15}+18=\\ 28-7\sqrt{15}.\end{array}$$

19.

$$\begin{array}{r}3\sqrt{7}+2\sqrt{5}\\ 2\sqrt{7}-3\sqrt{5}\\ \hline 42+4\sqrt{35}\qquad\\ -9\sqrt{35}-30\\ \hline 42-5\sqrt{35}-30=\\ 12-5\sqrt{35}.\end{array}$$

20.

$$\begin{array}{r}4\sqrt{3}+2\sqrt{5}\\ 4\sqrt{5}-3\sqrt{3}\\ \hline 16\sqrt{15}+40\\ -\ 6\sqrt{15}-36\\ \hline 10\sqrt{15}+\ 4.\end{array}$$

21.

$$\begin{array}{r}\sqrt[3]{2}+\sqrt[3]{3}\qquad\\ \sqrt[3]{2}+\sqrt[3]{3}\qquad\\ \hline \sqrt[3]{4}+\ \sqrt[3]{6}\qquad\\ +\ \sqrt[3]{6}+\sqrt[3]{9}\\ \hline \sqrt[3]{4}+2\sqrt[3]{6}+\sqrt[3]{9}.\end{array}$$

22.

$$\begin{array}{r}\sqrt[3]{5}+\sqrt[3]{7}\qquad\\ \sqrt[3]{5}-\sqrt[3]{7}\qquad\\ \hline \sqrt[3]{25}+\sqrt[3]{35}\qquad\\ -\sqrt[3]{35}-\sqrt[3]{49}\\ \hline \sqrt[3]{25}\qquad\quad -\sqrt[3]{49}.\end{array}$$

23.

$$\begin{array}{r}2\sqrt[3]{4}+3\sqrt[3]{6}\qquad\qquad\\ 3\sqrt[3]{4}-2\sqrt[3]{6}\qquad\qquad\\ \hline 6\sqrt[3]{16}+\ 9\sqrt[3]{24}\qquad\qquad\\ -\ 4\sqrt[3]{24}-6\sqrt[3]{36}\\ \hline 6\sqrt[3]{16}+\ 5\sqrt[3]{24}-6\sqrt[3]{36}=\\ 12\sqrt[3]{2}+10\sqrt[3]{3}-6\sqrt[3]{36}.\end{array}$$

24.

$$\begin{array}{r}a\sqrt{b}+x\sqrt{y}\qquad\\ a\sqrt{b}-x\sqrt{y}\qquad\\ \hline a^2b+ax\sqrt{by}\qquad\\ -ax\sqrt{by}-x^2y\\ \hline a^2b\qquad\quad -x^2y.\end{array}$$

25

$$\begin{array}{r}a\sqrt[3]{b^2}+x\sqrt[3]{y^2}\qquad\qquad\\ a\sqrt[3]{b^2}-x\sqrt[3]{y^2}\qquad\qquad\\ \hline a^2b\sqrt[3]{b}+ax\sqrt[3]{b^2y^2}\qquad\qquad\\ -ax\sqrt[3]{b^2y^2}-x^2y\sqrt[3]{y}\\ \hline a^2b\sqrt[3]{b}\qquad\qquad -x^2y\sqrt[3]{y}.\end{array}$$

26.

$$
\begin{array}{l}
\sqrt{2}+\sqrt{3}-\sqrt{5} \\
\underline{\sqrt{2}-\sqrt{3}+\sqrt{5}} \\
\quad 2+\sqrt{6}-\sqrt{10} \\
-3-\sqrt{6} \qquad +\sqrt{15} \\
\underline{-5 \qquad +\sqrt{10}+\sqrt{15}} \\
-6 \qquad\qquad +2\sqrt{15}=2\sqrt{15}-6.
\end{array}
$$

27.

$$
\begin{array}{l}
\sqrt{3}-\sqrt{5}+2\sqrt{7} \\
\underline{3\sqrt{3}+2\sqrt{5}-\sqrt{7}} \\
\quad 9-3\sqrt{15}+6\sqrt{21} \\
-10+2\sqrt{15} \qquad +4\sqrt{35} \\
\underline{-14 \qquad -\sqrt{21}+\sqrt{35}} \\
-15-\sqrt{15}+5\sqrt{21}+5\sqrt{35}= \\
\qquad 5\sqrt{35}+5\sqrt{21}-\sqrt{15}-15.
\end{array}
$$

28.

$$
\begin{array}{l}
\sqrt{x+1}+\sqrt{x-1} \\
\underline{\sqrt{x+1}-\sqrt{x-1}} \\
(x+1)+\sqrt{x^2-1} \\
\underline{\qquad -\sqrt{x^2-1}-(x-1)} \\
(x+1) \qquad -(x-1)=2.
\end{array}
$$

29.

$$
\begin{array}{l}
\sqrt{2x+5}+\sqrt{2x-1} \\
\underline{\sqrt{2x+5}-\sqrt{2x-1}} \\
(2x+5)+\sqrt{4x^2+8x-5} \\
\underline{\qquad -\sqrt{4x^2+8x-5}-(2x-1)} \\
(2x+5) \qquad -(2x-1)=6.
\end{array}
$$

30.

$$
\begin{array}{l}
\sqrt[3]{3}+2\sqrt[3]{5}-\sqrt[3]{4} \\
\underline{\sqrt[3]{9}-\sqrt[3]{2}} \\
\quad 3+2\sqrt[3]{45}-\sqrt[3]{36} \\
\underline{+2 \qquad\qquad -\sqrt[3]{6}-2\sqrt[3]{10}} \\
\quad 5+2\sqrt[3]{45}-\sqrt[3]{36}-\sqrt[3]{6}-2\sqrt[3]{10}.
\end{array}
$$

31.

$$\begin{array}{llll}
\sqrt[3]{a}+\sqrt[3]{b}-\sqrt[3]{c} & & & \\
\sqrt[3]{a}-\sqrt[3]{b}+\sqrt[3]{c} & & & \\
\hline
\sqrt[3]{a^2}+\sqrt[3]{ab}-\sqrt[3]{ac} & & & \\
\quad -\sqrt[3]{ab} & -\sqrt[3]{b^2} & +\sqrt[3]{bc} & \\
\qquad +\sqrt[3]{ac} & & +\sqrt[3]{bc} & -\sqrt[3]{c^2} \\
\hline
\sqrt[3]{a^2} & -\sqrt[3]{b^2} & +2\sqrt[3]{bc} & -\sqrt[3]{c^2}.
\end{array}$$

32.

$$\begin{array}{lll}
\sqrt[3]{4}-\sqrt[3]{16}+\sqrt[3]{12} & & \\
\sqrt[3]{4}+\sqrt[3]{16}-\sqrt[3]{2} & & \\
\hline
2\sqrt[3]{2}-4+2\sqrt[3]{6} & & \\
\quad +4 & -4\sqrt[3]{4} & +4\sqrt[3]{3} \\
\quad -2 & +2\sqrt[3]{4} & -2\sqrt[3]{3} \\
\hline
2\sqrt[3]{2}-2+2\sqrt[3]{6} & -2\sqrt[3]{4} & +2\sqrt[3]{3}.
\end{array}$$

33.

$$\begin{array}{ll}
\sqrt[4]{8}-\sqrt[4]{27}+\sqrt[4]{24} & \\
\sqrt[4]{2}-\sqrt[4]{3} & \\
\hline
2-\sqrt[4]{54}+2\sqrt[4]{3} & \\
3 & -\sqrt[4]{24}-\sqrt[4]{72} \\
\hline
5-\sqrt[4]{54}+2\sqrt[4]{3} & -\sqrt[4]{24}-\sqrt[4]{72}.
\end{array}$$

Art. 293.

1. $$\left.\begin{matrix}\sqrt{3}\\ \sqrt[3]{2}\end{matrix}\right\} = \left\{\begin{matrix}3^{\frac{1}{2}}\\ 2^{\frac{1}{3}}\end{matrix}\right\} = \left\{\begin{matrix}3^{\frac{3}{6}}\\ 2^{\frac{2}{6}}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[6]{27}\\ \sqrt[6]{4}\end{matrix}\right\};$$

$$\sqrt[6]{27}\times\sqrt[6]{4} = \sqrt[6]{27\times 4} = \sqrt[6]{108}.$$

2. $$\left.\begin{matrix}\sqrt[3]{4}\\ \sqrt[4]{3}\end{matrix}\right\} = \left\{\begin{matrix}4^{\frac{1}{3}}\\ 3^{\frac{1}{4}}\end{matrix}\right\} = \left\{\begin{matrix}4^{\frac{4}{12}}\\ 3^{\frac{3}{12}}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[12]{256}\\ \sqrt[12]{27}\end{matrix}\right\};$$

$$\sqrt[12]{256}\times\sqrt[12]{27} = \sqrt[12]{256\times 27} = \sqrt[12]{6912}.$$

3. $\left.\begin{matrix}\sqrt[3]{2ax}\\ \sqrt[6]{2ax}\end{matrix}\right\} = \left\{\begin{matrix}(2ax)^{\frac{1}{3}}\\ (2ax)^{\frac{1}{6}}\end{matrix}\right\} = \left\{\begin{matrix}(2ax)^{\frac{2}{6}}\\ (2ax)^{\frac{1}{6}}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[6]{4a^2x^2}\\ \sqrt[6]{2ax}\end{matrix}\right\};$

$$\sqrt[6]{4a^2x^2} \times \sqrt[6]{2ax} = \sqrt[6]{4a^2x^2 \times 2ax} = \sqrt{2ax}.$$

4. $\left.\begin{matrix}3\sqrt{2ax}\\ 2\sqrt[3]{xy}\end{matrix}\right\} = \left\{\begin{matrix}3(2ax)^{\frac{1}{2}}\\ 2(xy)^{\frac{1}{3}}\end{matrix}\right\} = \left\{\begin{matrix}3(2ax)^{\frac{3}{6}}\\ 2(xy)^{\frac{2}{6}}\end{matrix}\right\} = \left\{\begin{matrix}3\sqrt[6]{8a^3x^3}\\ 2\sqrt[6]{x^2y^2}\end{matrix}\right\};$

$$3\sqrt[6]{8a^3x^3} \times 2\sqrt[6]{x^2y^2} = 6\sqrt[6]{8a^3x^5y^2}.$$

5. $\left.\begin{matrix}2\sqrt{\frac{1}{2}}\\ 3\sqrt[3]{2}\end{matrix}\right\} = \left\{\begin{matrix}2(\frac{1}{2})^{\frac{1}{2}}\\ 3(2)^{\frac{1}{3}}\end{matrix}\right\} = \left\{\begin{matrix}2(\frac{1}{2})^{\frac{3}{6}}\\ 3(2)^{\frac{2}{6}}\end{matrix}\right\} = \left\{\begin{matrix}2\sqrt[6]{\frac{1}{8}}\\ 3\sqrt[6]{4}\end{matrix}\right\};$

$$2\sqrt[6]{\tfrac{1}{8}} \times 3\sqrt[6]{4} = 6\sqrt[6]{\tfrac{1}{2} \times \tfrac{64}{64}} = 3\sqrt[6]{32}.$$

6. $\left.\begin{matrix}\sqrt[3]{24}\\ 6\sqrt{3}\end{matrix}\right\} = \left\{\begin{matrix}24^{\frac{1}{3}}\\ 6(3)^{\frac{1}{2}}\end{matrix}\right\} = \left\{\begin{matrix}24^{\frac{2}{6}}\\ 6(3)^{\frac{3}{6}}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[6]{576}\\ 6\sqrt[6]{3}\end{matrix}\right\};$

$$\sqrt[6]{576} \times 6\sqrt[6]{3} = 6\sqrt[6]{3^3 \times 2^6} = 12\sqrt{3}.$$

7. $\left.\begin{matrix}\sqrt{2}\\ \sqrt[3]{3}\end{matrix}\right\} = \left\{\begin{matrix}2^{\frac{1}{2}}\\ 3^{\frac{1}{3}}\end{matrix}\right\} = \left\{\begin{matrix}2^{\frac{3}{6}}\\ 3^{\frac{2}{6}}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[6]{8}\\ \sqrt[6]{9}\end{matrix}\right\};$

$$\sqrt[6]{8} \times \sqrt[6]{9} = \sqrt[6]{72}.$$

8. $\left.\begin{matrix}\sqrt{5}\\ \sqrt[3]{4}\end{matrix}\right\} = \left\{\begin{matrix}5^{\frac{1}{2}}\\ 4^{\frac{1}{3}}\end{matrix}\right\} = \left\{\begin{matrix}5^{\frac{3}{6}}\\ 4^{\frac{2}{6}}\end{matrix}\right\} = \left\{\begin{matrix}\sqrt[6]{125}\\ \sqrt[6]{16}\end{matrix}\right\};$

$$\sqrt[6]{125} \times \sqrt[6]{16} = \sqrt[6]{2000}.$$

9. $\left.\begin{matrix}2\sqrt[3]{6}\\ 4\sqrt{3}\end{matrix}\right\} = \left\{\begin{matrix}2(6)^{\frac{1}{3}}\\ 4(3)^{\frac{1}{2}}\end{matrix}\right\} = \left\{\begin{matrix}2(6)^{\frac{2}{6}}\\ 4(3)^{\frac{3}{6}}\end{matrix}\right\} = \left\{\begin{matrix}2\sqrt[6]{36}\\ 4\sqrt[6]{27}\end{matrix}\right\};$

$$2\sqrt[6]{36} \times 4\sqrt[6]{27} = 8\sqrt[6]{972}.$$

10. $\left.\begin{matrix} 5\sqrt{3} \\ 3\sqrt[4]{27} \end{matrix}\right\} = \left\{\begin{matrix} 5(3)^{\frac{1}{2}} \\ 3(27)^{\frac{1}{4}} \end{matrix}\right\} = \left\{\begin{matrix} 5(3)^{\frac{2}{4}} \\ 3(27)^{\frac{1}{4}} \end{matrix}\right\} = \left\{\begin{matrix} 5\sqrt[4]{9} \\ 3\sqrt[4]{27} \end{matrix}\right\};$

$$5\sqrt[4]{9} \times 3\sqrt[4]{27} = 15\sqrt[4]{3^4 \times 3} = 45\sqrt[4]{3}.$$

11. $\left.\begin{matrix} \sqrt[3]{9} \\ \sqrt[6]{18} \end{matrix}\right\} = \left\{\begin{matrix} 9^{\frac{1}{3}} \\ 18^{\frac{1}{6}} \end{matrix}\right\} = \left\{\begin{matrix} 9^{\frac{2}{6}} \\ 18^{\frac{1}{6}} \end{matrix}\right\} = \left\{\begin{matrix} \sqrt[6]{81} \\ \sqrt[6]{18} \end{matrix}\right\};$

$$\sqrt[6]{81} \times \sqrt[6]{18} = \sqrt[6]{3^6 \times 2} = 3\sqrt[6]{2}.$$

12. $\left.\begin{matrix} 3\sqrt[3]{25} \\ \sqrt{15} \end{matrix}\right\} = \left\{\begin{matrix} 3(25)^{\frac{1}{3}} \\ (15)^{\frac{1}{2}} \end{matrix}\right\} = \left\{\begin{matrix} 3(25)^{\frac{2}{6}} \\ (15)^{\frac{3}{6}} \end{matrix}\right\} = \left\{\begin{matrix} 3\sqrt[6]{5^4} \\ \sqrt[6]{5^2 \times 135} \end{matrix}\right\};$

$$3\sqrt[6]{5^4} \times \sqrt[6]{5^2 \times 135} = 15\sqrt[6]{135}.$$

13. $\left.\begin{matrix} a\sqrt{b} \\ b\sqrt[3]{c} \end{matrix}\right\} = \left\{\begin{matrix} a(b)^{\frac{1}{2}} \\ b(c)^{\frac{1}{3}} \end{matrix}\right\} = \left\{\begin{matrix} a(b)^{\frac{3}{6}} \\ b(c)^{\frac{2}{6}} \end{matrix}\right\} = \left\{\begin{matrix} a\sqrt[6]{b^3} \\ b\sqrt[6]{c^2} \end{matrix}\right\};$

$$a\sqrt[6]{b^3} \times b\sqrt[6]{c^2} = ab\sqrt[6]{b^3c^2}.$$

14. $\left.\begin{matrix} 8\sqrt{9} \\ 2\sqrt[3]{3} \end{matrix}\right\} = \left\{\begin{matrix} 8(9)^{\frac{1}{2}} \\ 2(3)^{\frac{1}{3}} \end{matrix}\right\} = \left\{\begin{matrix} 8(9)^{\frac{3}{6}} \\ 2(3)^{\frac{2}{6}} \end{matrix}\right\} = \left\{\begin{matrix} 8\sqrt[6]{9^3} \\ 2\sqrt[6]{3^2} \end{matrix}\right\};$

$$8\sqrt[6]{9^3} \times 2\sqrt[6]{3^2} = 16\sqrt[6]{6561} = 48\sqrt{3}.$$

15. $\left.\begin{matrix} \sqrt[3]{a-x} \\ \sqrt{a-x} \end{matrix}\right\} = \left\{\begin{matrix} (a-x)^{\frac{1}{3}} \\ (a-x)^{\frac{1}{2}} \end{matrix}\right\} = \left\{\begin{matrix} (a-x)^{\frac{2}{6}} \\ (a-x)^{\frac{3}{6}} \end{matrix}\right\} = \left\{\begin{matrix} \sqrt[6]{(a-x)^2} \\ \sqrt[6]{(a-x)^3} \end{matrix}\right\};$

$$\sqrt[6]{(a-x)^2} \times \sqrt[6]{(a-x)^3} = \sqrt[6]{(a-x)^5}.$$

16. $\left.\begin{matrix} \sqrt[5]{\frac{3}{7}} \\ \sqrt[3]{\frac{7}{3}} \end{matrix}\right\} = \left\{\begin{matrix} (\frac{3}{7})^{\frac{1}{5}} \\ (\frac{7}{3})^{\frac{1}{3}} \end{matrix}\right\} = \left\{\begin{matrix} (\frac{3}{7})^{\frac{3}{15}} \\ (\frac{7}{3})^{\frac{5}{15}} \end{matrix}\right\} = \left\{\begin{matrix} \sqrt[15]{(\frac{3}{7})^3} \\ \sqrt[15]{(\frac{7}{3})^5} \end{matrix}\right\};$

$$\sqrt[15]{(\tfrac{3}{7})^3} \times \sqrt[15]{(\tfrac{7}{3})^5} = \sqrt[15]{(\tfrac{7}{3})^2} = \sqrt[15]{(\tfrac{7}{3})^2 \times (\tfrac{3}{3})^{18}} = \tfrac{1}{3}\sqrt[15]{49 \times 3^{13}}.$$

17. $$\left.\begin{array}{l}\sqrt{\frac{2}{5}}\\ \sqrt[3]{\frac{3}{8}}\end{array}\right\}=\left\{\begin{array}{l}(\frac{2}{5})^{\frac{1}{2}}\\ (\frac{3}{8})^{\frac{1}{3}}\end{array}\right\}=\left\{\begin{array}{l}(\frac{2}{5})^{\frac{3}{6}}\\ (\frac{3}{8})^{\frac{2}{6}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[6]{(\frac{2}{5})^3}\\ \sqrt[6]{(\frac{3}{8})^2}\end{array}\right\};$$

$$\sqrt[6]{(\tfrac{2}{5})^3}\times\sqrt[6]{(\tfrac{3}{8})^2}=\sqrt[6]{\tfrac{72}{8000}\times\tfrac{125}{125}}=\tfrac{1}{10}\sqrt[6]{9000}.$$

18. $$\left.\begin{array}{l}\sqrt{\frac{2}{3}}\\ \sqrt[3]{\frac{3}{2}}\end{array}\right\}=\left\{\begin{array}{l}(\frac{2}{3})^{\frac{1}{2}}\\ (\frac{3}{2})^{\frac{1}{3}}\end{array}\right\}=\left\{\begin{array}{l}(\frac{2}{3})^{\frac{3}{6}}\\ (\frac{3}{2})^{\frac{2}{6}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[6]{(\frac{2}{3})^3}\\ \sqrt[6]{(\frac{3}{2})^2}\end{array}\right\};$$

$$\sqrt[6]{(\tfrac{2}{3})^3}\times\sqrt[6]{(\tfrac{3}{2})^2}=\sqrt[6]{\tfrac{2}{3}\times(\tfrac{3}{3})^5}=\tfrac{1}{3}\sqrt[6]{486}.$$

19. $$\left.\begin{array}{l}4\sqrt{ab}\\ 5\sqrt[3]{ab^2}\end{array}\right\}=\left\{\begin{array}{l}4(ab)^{\frac{1}{2}}\\ 5(ab^2)^{\frac{1}{3}}\end{array}\right\}=\left\{\begin{array}{l}4(ab)^{\frac{3}{6}}\\ 5(ab^2)^{\frac{2}{6}}\end{array}\right\}=\left\{\begin{array}{l}4\sqrt[6]{a^3b^3}\\ 5\sqrt[6]{a^2b^4}\end{array}\right\};$$

$$4\sqrt[6]{a^3b^3}\times5\sqrt[6]{a^2b^4}=20\sqrt[6]{a^5b^7}=20b\sqrt[6]{a^5b}.$$

20. $$\left.\begin{array}{l}\sqrt[3]{9x^2}\\ \sqrt{3x}\end{array}\right\}=\left\{\begin{array}{l}(9x^2)^{\frac{1}{3}}\\ (3x)^{\frac{1}{2}}\end{array}\right\}=\left\{\begin{array}{l}(9x^2)^{\frac{2}{6}}\\ (3x)^{\frac{3}{6}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[6]{3^4x^4}\\ \sqrt[6]{3^3x^3}\end{array}\right\};$$

$$\sqrt[6]{3^4x^4}\times\sqrt[6]{3^3x^3}=\sqrt[6]{3^7x^7}=3x\sqrt[6]{3x}.$$

21. $$\left.\begin{array}{l}\frac{1}{2}\sqrt{\frac{1}{2}}\\ \frac{2}{3}\sqrt[3]{\frac{2}{3}}\end{array}\right\}=\left\{\begin{array}{l}\frac{1}{2}(\frac{1}{2})^{\frac{1}{2}}\\ \frac{2}{3}(\frac{2}{3})^{\frac{1}{3}}\end{array}\right\}=\left\{\begin{array}{l}\frac{1}{2}(\frac{1}{2})^{\frac{3}{6}}\\ \frac{2}{3}(\frac{2}{3})^{\frac{2}{6}}\end{array}\right\}=\left\{\begin{array}{l}\frac{1}{2}\sqrt[6]{\frac{1}{8}}\\ \frac{2}{3}\sqrt[6]{\frac{4}{9}}\end{array}\right\};$$

$$\tfrac{1}{2}\sqrt[6]{\tfrac{1}{8}}\times\tfrac{2}{3}\sqrt[6]{\tfrac{4}{9}}=\tfrac{1}{3}\sqrt[6]{\tfrac{1}{18}\times\tfrac{2592}{2592}}=\tfrac{1}{18}\sqrt[6]{2592}.$$

22. $$\left.\begin{array}{l}\frac{2}{3}\sqrt[3]{4}\\ \frac{3}{4}\sqrt{6}\end{array}\right\}=\left\{\begin{array}{l}\frac{2}{3}(4)^{\frac{1}{3}}\\ \frac{3}{4}(6)^{\frac{1}{2}}\end{array}\right\}=\left\{\begin{array}{l}\frac{2}{3}(4)^{\frac{2}{6}}\\ \frac{3}{4}(6)^{\frac{3}{6}}\end{array}\right\}=\left\{\begin{array}{l}\frac{2}{3}\sqrt[6]{16}\\ \frac{3}{4}\sqrt[6]{216}\end{array}\right\};$$

$$\tfrac{2}{3}\sqrt[6]{2^4}\times\tfrac{3}{4}\sqrt[6]{2^3\times3^3}=\tfrac{1}{2}\sqrt[6]{2^7\times3^3}=\sqrt[6]{54}.$$

23. $$\left.\begin{array}{l}\sqrt[5]{a^2x}\\ 3\sqrt{2a^3x}\end{array}\right\}=\left\{\begin{array}{l}(a^2x)^{\frac{1}{5}}\\ 3(2a^3x)^{\frac{1}{2}}\end{array}\right\}=\left\{\begin{array}{l}(a^2x)^{\frac{2}{10}}\\ 3(2a^3x)^{\frac{5}{10}}\end{array}\right\}=\left\{\begin{array}{l}\sqrt[10]{a^4x^2}\\ 3\sqrt[10]{32a^{15}x^5}\end{array}\right\};$$

$$\sqrt[10]{a^4x^2}\times3\sqrt[10]{32a^{15}x^5}=3a\sqrt[10]{32a^9x^7}.$$

24. $$\left.\begin{matrix}\sqrt[3]{ab^2}\\ \sqrt{ab}\end{matrix}\right\} = \begin{Bmatrix}(ab^2)^{\frac{1}{3}}\\ (ab)^{\frac{1}{2}}\end{Bmatrix} = \begin{Bmatrix}(ab^2)^{\frac{2}{6}}\\ (ab)^{\frac{3}{6}}\end{Bmatrix} = \begin{Bmatrix}\sqrt[6]{a^2b^4}\\ \sqrt[6]{a^3b^3}\end{Bmatrix};$$

$$\sqrt[6]{a^2b^4} \times \sqrt[6]{a^3b^3} = b\sqrt[6]{a^5b}.$$

25. $$\left.\begin{matrix}3\sqrt[3]{4a^2}\\ 4\sqrt{2a}\end{matrix}\right\} = \begin{Bmatrix}3\,(4a^2)^{\frac{1}{3}}\\ 4\,(2a)^{\frac{1}{2}}\end{Bmatrix} = \begin{Bmatrix}3\,(4a^2)^{\frac{2}{6}}\\ 3\,(2a)^{\frac{3}{6}}\end{Bmatrix} = \begin{Bmatrix}3\sqrt[6]{16a^4}\\ 4\sqrt[6]{8a^3}\end{Bmatrix};$$

$$3\sqrt[6]{16a^4} \times 4\sqrt[6]{8a^3} = 12\sqrt[6]{128a^7} = 24a\sqrt[6]{2a}.$$

26. $$\left.\begin{matrix}5\sqrt{12a}\\ 3\sqrt[3]{16a^2}\end{matrix}\right\} = \begin{Bmatrix}5\,(12a)^{\frac{1}{2}}\\ 3\,(16a^2)^{\frac{1}{3}}\end{Bmatrix} = \begin{Bmatrix}5\,(12a)^{\frac{3}{6}}\\ 3\,(16a^2)^{\frac{2}{6}}\end{Bmatrix} = \begin{Bmatrix}5\sqrt[6]{(12a)^3}\\ 3\sqrt[6]{(16a^2)^2}\end{Bmatrix};$$

$$5\sqrt[6]{(12a)^3} \times 3\sqrt[6]{(16a^2)^2} = 60a\sqrt[6]{108a}.$$

27. $$\left.\begin{matrix}\frac{1}{2}\sqrt{5}\\ \frac{1}{3}\sqrt[4]{10}\end{matrix}\right\} = \begin{Bmatrix}\frac{1}{2}(5)^{\frac{1}{2}}\\ \frac{1}{3}(10)^{\frac{1}{4}}\end{Bmatrix} = \begin{Bmatrix}\frac{1}{2}(5)^{\frac{2}{4}}\\ \frac{1}{3}(10)^{\frac{1}{4}}\end{Bmatrix} = \begin{Bmatrix}\frac{1}{2}\sqrt[4]{25}\\ \frac{1}{3}\sqrt[4]{10}\end{Bmatrix};$$

$$\frac{1}{2}\sqrt[4]{25} \times \frac{1}{3}\sqrt[4]{10} = \frac{1}{6}\sqrt[4]{250}.$$

28. $$\left.\begin{matrix}3\sqrt{x}\\ \sqrt[6]{y}\\ 2\sqrt[4]{x}\\ \sqrt[6]{y}\end{matrix}\right\} = \begin{Bmatrix}6\,(x)^{\frac{3}{4}}\\ (y)^{\frac{2}{6}}\end{Bmatrix} = \begin{Bmatrix}6\,(x)^{\frac{9}{12}}\\ (y)^{\frac{4}{12}}\end{Bmatrix} = \begin{Bmatrix}6\sqrt[12]{x^9}\\ \sqrt[12]{y^4}\end{Bmatrix};$$

$$6\sqrt[12]{x^9} \times \sqrt[12]{y^4} = 6\sqrt[12]{x^9y^4}.$$

29. $$\left.\begin{matrix}3\sqrt[3]{\frac{1}{2}}\\ 2\sqrt{\frac{2}{3}}\\ 5\sqrt[6]{108}\end{matrix}\right\} = \begin{Bmatrix}3\,(\frac{1}{2})^{\frac{1}{3}}\\ 2\,(\frac{2}{3})^{\frac{1}{2}}\\ 5\,(108)^{\frac{1}{6}}\end{Bmatrix} = \begin{Bmatrix}3\,(\frac{1}{2})^{\frac{2}{6}}\\ 2\,(\frac{2}{3})^{\frac{3}{6}}\\ 5\,(108)^{\frac{1}{6}}\end{Bmatrix} = \begin{Bmatrix}3\sqrt[6]{\frac{1}{4}}\\ 2\sqrt[6]{\frac{8}{27}}\\ 5\sqrt[6]{108}\end{Bmatrix};$$

$$3\sqrt[6]{\tfrac{1}{4}} \times 2\sqrt[6]{\tfrac{8}{27}} \times 5\sqrt[6]{108} = 30\sqrt{2}.$$

30. $$\left.\begin{array}{r} 3\sqrt{\tfrac{2}{3}} \\ 4\sqrt[3]{\tfrac{3}{4}} \\ \sqrt[4]{2} \end{array}\right\} = \left\{\begin{array}{c} 3(\tfrac{2}{3})^{\frac{1}{2}} \\ 4(\tfrac{3}{4})^{\frac{1}{3}} \\ (2)^{\frac{1}{4}} \end{array}\right\} = \left\{\begin{array}{c} 3(\tfrac{2}{3})^{\frac{6}{12}} \\ 4(\tfrac{3}{4})^{\frac{4}{12}} \\ (2)^{\frac{3}{12}} \end{array}\right\} = \left\{\begin{array}{c} 3\sqrt[12]{(\tfrac{2}{3})^6} \\ 4\sqrt[12]{(\tfrac{3}{4})^4} \\ \sqrt[12]{2^3} \end{array}\right\};$$

$$3\sqrt[12]{(\tfrac{2}{3})^6} \times 4\sqrt[12]{(\tfrac{3}{4})^4} \times \sqrt[12]{2^3} = 12\sqrt[12]{\tfrac{2}{9}} = 4\sqrt[12]{2 \times 3^{10}}.$$

31. $$\begin{array}{rr} \sqrt[3]{a} - \sqrt[6]{b^4} & \\ \sqrt[6]{a^2} + \sqrt[3]{b^2} & \\ \hline \sqrt[3]{a} \times \sqrt[6]{a^2} - \sqrt[3]{ab^2} & \\ + \sqrt[3]{ab^2} & - \sqrt[3]{b^2} \times \sqrt[6]{b^4} \\ \hline \sqrt[3]{a^2} & - b\sqrt[3]{b}. \end{array}$$

32. $$\left.\begin{array}{r} 3\sqrt{2ax} \\ 2\sqrt[5]{3a^2x^2} \end{array}\right\} = \left\{\begin{array}{l} 3(2ax)^{\frac{1}{2}} \\ 2(3a^2x^2)^{\frac{1}{5}} \end{array}\right\} = \left\{\begin{array}{l} 3(2ax)^{\frac{5}{10}} \\ 2(3a^2x^2)^{\frac{2}{10}} \end{array}\right\} = \left\{\begin{array}{l} 3\sqrt[10]{32a^5x^5} \\ 2\sqrt[10]{9a^4x^4} \end{array}\right\};$$

$$3\sqrt[10]{32a^5x^5} \times 2\sqrt[10]{9a^4x^4} = 6a\sqrt[10]{288ax^9}.$$

33. $$\begin{array}{rr} \sqrt[3]{2} - \sqrt{5} & \\ \sqrt[3]{2} + \sqrt{5} & \\ \hline \sqrt[3]{4} - \sqrt[3]{2} \times \sqrt{5} & \\ + \sqrt[3]{2} \times \sqrt{5} & - 5 \\ \hline \sqrt[3]{4} & - 5. \end{array}$$

34. $$\begin{array}{rr} \sqrt[3]{4} + \sqrt{3} & \\ \sqrt[3]{4} + \sqrt{3} & \\ \hline \sqrt[3]{16} + \sqrt{3} \times \sqrt[3]{4} & \\ \sqrt{3} \times \sqrt[3]{4} & + 3 \\ \hline 2\sqrt[3]{2} + \quad 2\sqrt[6]{432} & + 3. \end{array}$$

35. $$\begin{array}{rr} 2\sqrt[3]{2} + 3\sqrt{3} & \\ 5\sqrt[3]{2} - 2\sqrt{3} & \\ \hline 10\sqrt[3]{4} + 15\sqrt[3]{2} \times \sqrt{3} & \\ - 4\sqrt[3]{2} \times \sqrt{3} & - 18 \\ \hline 10\sqrt[3]{4} + 11\sqrt[6]{108} & - 18. \end{array}$$

36. $$\begin{array}{rr} 3\sqrt{7} + \sqrt[3]{5} & \\ \sqrt[3]{7} - \sqrt{5} & \\ \hline 3\sqrt[6]{7^5} + \sqrt[3]{35} & \\ & - 3\sqrt{35} - \sqrt[6]{5^5} \\ \hline 3\sqrt[6]{7^5} + \sqrt[3]{35} & - 3\sqrt{35} - \sqrt[6]{5^5}. \end{array}$$

37.

$$\begin{array}{l}
\sqrt{2}-\sqrt[3]{3}+\sqrt{5} \\
\sqrt[3]{2}-\sqrt{3} \\
\hline
\sqrt[3]{2}\times\sqrt{2}-\sqrt[3]{6}+\sqrt[3]{2}\times\sqrt{5} \\
\qquad -\sqrt{6}+\sqrt{3}\times\sqrt[3]{3}-\sqrt{15} \\
\hline
\sqrt[6]{32}-\sqrt[3]{6}+\sqrt[6]{500}-\sqrt{6}+\sqrt[6]{243}-\sqrt{15}.
\end{array}$$

38.

$$\begin{array}{l}
\sqrt[3]{2}+\sqrt[4]{4} \\
\sqrt[3]{2}-\sqrt[4]{4} \\
\hline
\sqrt[3]{4}+\sqrt[3]{2}\times\sqrt[4]{4} \\
\qquad -\sqrt[3]{2}\times\sqrt[4]{4}-2 \\
\hline
\sqrt[3]{4}\qquad -2.
\end{array}$$

39.

$$\begin{array}{l}
\sqrt{2}+\sqrt[5]{3}-\sqrt[10]{4} \\
\sqrt[5]{2}-\sqrt{3} \\
\hline
\sqrt{2}\times\sqrt[5]{2}+\sqrt[5]{6}-\sqrt[5]{2}\times\sqrt[10]{4} \\
\qquad -\sqrt{6}-\sqrt{3}\times\sqrt[5]{3}+\sqrt{3}\times\sqrt[10]{4} \\
\hline
\sqrt[10]{128}+\sqrt[5]{6}-\sqrt[5]{4}-\sqrt{6}-\sqrt[10]{2187}+\sqrt[10]{972}.
\end{array}$$

40.

$$\begin{array}{l}
\sqrt[3]{a}+\sqrt[4]{b}-\sqrt[6]{c} \\
\sqrt{a}-\sqrt[12]{b} \\
\hline
\sqrt[3]{a}\times\sqrt{a}+\sqrt{a}\times\sqrt[4]{b}-\sqrt{a}\times\sqrt[6]{c} \\
\qquad -\sqrt[3]{a}\times\sqrt[12]{b}-\sqrt[4]{b}\times\sqrt[12]{b}+\sqrt[12]{b}\times\sqrt[6]{c} \\
\hline
\sqrt[6]{a^5}+\sqrt[4]{a^2b}-\sqrt[6]{a^3c}-\sqrt[12]{a^4b}-\sqrt[3]{b}+\sqrt[12]{bc^2}.
\end{array}$$

41.

$$\begin{array}{l}
2\sqrt{2}-5\sqrt[5]{3} \\
3\sqrt[5]{2}-\sqrt[10]{10} \\
\hline
6\sqrt{2}\times\sqrt[5]{2}-15\sqrt[5]{6} \\
\qquad -2\sqrt{2}\times\sqrt[10]{10}+5\sqrt[5]{3}\times\sqrt[10]{10} \\
\hline
6\sqrt[10]{128}-15\sqrt[5]{6}-2\sqrt[10]{320}+5\sqrt[10]{90}.
\end{array}$$

42.

$$\begin{array}{l}
8\sqrt[3]{a^2}-2\sqrt[4]{a^3} \\
2\sqrt[12]{x^4}+3\sqrt[6]{y^3} \\
\hline
6\sqrt[3]{a^2}\times\sqrt[12]{x^4}-4\sqrt[4]{a^3}\times\sqrt[12]{x^4} \\
\qquad\qquad +9\sqrt[3]{a^2}\times\sqrt[6]{y^3}-6\sqrt[4]{a^3}\times\sqrt[6]{y^3} \\
\hline
6\sqrt[3]{a^2x}-4\sqrt[12]{a^9x^4} \quad +9\sqrt[6]{a^4y^3} \quad -6\sqrt[4]{a^3y^2}.
\end{array}$$

43.

$$\begin{array}{l}
6\sqrt[6]{8}+5\sqrt[9]{27} \\
4\sqrt[12]{64}-3\sqrt[8]{81} \\
\hline
24\sqrt[6]{8}\times\sqrt[12]{64}+20\sqrt[9]{27}\times\sqrt[12]{64} \\
\qquad\qquad -18\sqrt[6]{8}\times\sqrt[8]{81}-15\sqrt[9]{27}\times\sqrt[8]{81} \\
\hline
48+20\sqrt[6]{72} \quad -18\sqrt{6} \quad -15\sqrt[6]{243}.
\end{array}$$

44.

$$\sqrt[10]{32}+\sqrt[6]{16}+\sqrt[9]{125} = \sqrt{2}+\sqrt[3]{4}+\sqrt[3]{5},$$

$$\sqrt[4]{4}-\sqrt[6]{25} = \sqrt{2}-\sqrt[3]{5},$$

$$\begin{array}{l}
\sqrt{2}+\sqrt[3]{4}+\sqrt[3]{5} \\
\sqrt{2}-\sqrt[3]{5} \\
\hline
2+\sqrt{2}\times\sqrt[3]{4}+\sqrt{2}\times\sqrt[3]{5} \\
\qquad\qquad -\sqrt{2}\times\sqrt[3]{5}-\sqrt[3]{20}-\sqrt[3]{25} \\
\hline
2+2\sqrt[6]{2} \qquad\qquad -\sqrt[3]{20}-\sqrt[3]{25}.
\end{array}$$

45.

$$\sqrt{2}-\sqrt[6]{9}+\sqrt[4]{25} = \sqrt{2}-\sqrt[3]{3}+\sqrt{5},$$

$$\sqrt[3]{3}+\sqrt[10]{32} = \sqrt[3]{3}+\sqrt{2}.$$

$$\begin{array}{l}
\sqrt{2}-\sqrt[3]{3}+\sqrt{5} \\
\sqrt[3]{3}+\sqrt{2} \\
\hline
\sqrt{2}\times\sqrt[3]{3}-\sqrt[3]{9}+\sqrt{5}\times\sqrt[3]{3} \\
-\sqrt{2}\times\sqrt[3]{3} \qquad\qquad +2+\sqrt{10} \\
\hline
\qquad -\sqrt[3]{9}+\sqrt[6]{1125} \quad +2+\sqrt{10}.
\end{array}$$

46.

$$\begin{array}{l}
2\sqrt{a}-3\sqrt[3]{b}+5\sqrt{c} \\
\sqrt[3]{a}-\sqrt{b} \\
\hline
2\sqrt{a}\times\sqrt[3]{a}-3\sqrt[3]{ab}+\sqrt[3]{a}\times 5\sqrt{c} \\
\qquad\qquad\qquad -2\sqrt{ab}+\sqrt{b}\times 3\sqrt[3]{b}+5\sqrt{bc} \\
\hline
2\sqrt[6]{a^5}\quad -3\sqrt[3]{ab}+5\sqrt[6]{a^2c^3}\quad -2\sqrt{ab}+3\sqrt[6]{b^5}\quad -5\sqrt{bc}.
\end{array}$$

47.

$$\begin{array}{l}
\sqrt{2}-\sqrt[3]{3}+\sqrt{5} \\
\sqrt{2}-\sqrt[3]{3}-\sqrt{5} \\
\hline
2-\sqrt[6]{72}+\sqrt{10} \\
\qquad\qquad +\sqrt[3]{9}-\sqrt[6]{1125} \\
-5-\sqrt[6]{72}-\sqrt{10}\qquad +\sqrt[6]{1125} \\
\hline
-3-2\sqrt[6]{72}\qquad +\sqrt[3]{9}.
\end{array}$$

48.

$$\begin{array}{l}
\sqrt{a}-\sqrt[4]{ab}+\sqrt{b} \\
\sqrt[4]{a}+\sqrt[4]{b} \\
\hline
\sqrt[4]{a^3}-\sqrt[4]{a^2b}+\sqrt[4]{ab^2} \\
\qquad +\sqrt[4]{a^2b}-\sqrt[4]{ab^2}+\sqrt[4]{b^3} \\
\hline
\sqrt[4]{a^3}\qquad\qquad +\sqrt[4]{b^3}.
\end{array}$$

49.

$$\begin{array}{l}
\sqrt{2}+\sqrt[4]{6}+\sqrt{3} \\
\sqrt{2}-\sqrt[4]{6}+\sqrt{3} \\
\hline
2+\sqrt[4]{24}+\sqrt{6} \\
\quad -\sqrt[4]{24}-\sqrt{6}-\sqrt[4]{54} \\
3\qquad +\sqrt{6}+\sqrt[4]{54} \\
\hline
5\qquad +\sqrt{6}
\end{array}$$

50.

$$\begin{array}{l}
\sqrt[3]{a}+\sqrt[6]{ab}+\sqrt[3]{b} \\
\sqrt[3]{a}-\sqrt[6]{ab}+\sqrt[3]{b} \\
\hline
\sqrt[3]{a^2}+\sqrt[6]{a^3b}+\sqrt[3]{ab} \\
\quad -\sqrt[6]{a^3b}-\sqrt[3]{ab}-\sqrt[6]{ab^3} \\
\qquad\qquad +\sqrt[3]{ab}+\sqrt[6]{ab^3}+\sqrt[3]{b^2} \\
\hline
\sqrt[3]{a^2}\qquad +\sqrt[3]{ab}\qquad +\sqrt[3]{b^2}.
\end{array}$$

51.

$$\begin{array}{l}
\sqrt[3]{x}+\sqrt[6]{xy}+\sqrt[3]{y} \\
\underline{\sqrt[6]{x}-\sqrt[6]{y}\qquad\qquad} \\
\sqrt{x}+\sqrt[6]{x^2y}+\sqrt[6]{xy^2} \\
\underline{\qquad -\sqrt[6]{x^2y}-\sqrt[6]{xy^2}-\sqrt{y}} \\
\sqrt{x}\qquad\qquad\qquad\qquad -\sqrt{y}.
\end{array}$$

52.

$$\begin{array}{l}
\sqrt[4]{5}+\sqrt[8]{15}+\sqrt[4]{3} \\
\underline{\sqrt[8]{5}-\sqrt[8]{3}\qquad\qquad} \\
\sqrt[8]{125}+\sqrt[8]{75}+\sqrt[8]{45} \\
\underline{\qquad -\sqrt[8]{75}-\sqrt[8]{45}-\sqrt[8]{27}} \\
\sqrt[8]{125}\qquad\qquad\qquad\qquad -\sqrt[8]{27}.
\end{array}$$

53.

$$\begin{array}{l}
2\sqrt{x}-5\sqrt[4]{xy}+3\sqrt{y} \\
\underline{\sqrt{x}+\tfrac{5}{2}\sqrt[4]{xy}-\tfrac{19}{4}\sqrt{y}} \\
2x-5\sqrt[4]{x^3y}+\ 3\sqrt{xy} \\
\qquad +5\sqrt[4]{x^3y}-\tfrac{25}{2}\sqrt{xy}+\tfrac{15}{2}\sqrt[4]{xy^3} \\
\underline{\qquad\qquad\qquad +\tfrac{19}{2}\sqrt{xy}-\tfrac{95}{4}\sqrt[4]{xy^3}+\tfrac{57}{4}y} \\
2x\qquad\qquad\qquad\qquad\qquad -\tfrac{65}{4}\sqrt[4]{xy^3}+\tfrac{57}{4}y.
\end{array}$$

54.

$$\begin{array}{l}
\sqrt[3]{x}-3\sqrt[6]{xy}-2\sqrt[3]{y} \\
\underline{\sqrt[3]{x}+3\sqrt[6]{xy}-2\sqrt[3]{y}} \\
\sqrt[3]{x^2}-3\sqrt[6]{x^3y}-\ 2\sqrt[3]{xy} \\
\qquad +3\sqrt[6]{x^3y}-\ 9\sqrt[3]{xy}-6\sqrt[6]{xy^3} \\
\underline{\qquad\qquad\qquad -\ 2\sqrt[3]{xy}+6\sqrt[6]{xy^3}+4\sqrt[3]{y^2}} \\
\sqrt[3]{x^2}\qquad\qquad -13\sqrt[3]{xy}\qquad\qquad +4\sqrt[3]{y^2}.
\end{array}$$

Art. 295.

1. $\dfrac{\sqrt{160}}{2\sqrt{8}}=\frac{1}{2}\sqrt{\frac{160}{8}}=\frac{1}{2}\sqrt{20}=\sqrt{5}.$

2. $\dfrac{24\sqrt{2}}{2\sqrt{8}}=12\sqrt{\frac{1}{4}}=\frac{12}{2}=6.$

3. $\dfrac{10\sqrt{72}}{5\sqrt{6}} = 2\sqrt{12} = 4\sqrt{3}.$

4. $\dfrac{\sqrt{125a^3x^2y}}{\sqrt{5a^2xy}} = \sqrt{\dfrac{125a^3x^2y}{5a^2xy}} = \sqrt{25ax} = 5\sqrt{ax}.$

5. $\dfrac{6\sqrt{10}}{\sqrt{15}} = 6\sqrt{\tfrac{2}{3}} = 6\sqrt{\tfrac{2}{3}\times\tfrac{3}{3}} = 2\sqrt{6}.$

6. $\dfrac{\sqrt{a^4b^2c}}{\sqrt[3]{ab}} = \sqrt[3]{a^3bc} = a\sqrt[3]{bc}.$

7. $\dfrac{\sqrt[3]{a^2-b^2}}{\sqrt[3]{a-b}} = \sqrt[3]{a+b}.$

8. $\dfrac{\sqrt{a^2-x^2}}{a-x} = \dfrac{1}{a-x}\sqrt{a^2-x^2}.$

9. $\dfrac{3\sqrt[4]{20}}{\sqrt[4]{5}} = 3\sqrt[4]{4} = 3\sqrt{2}.$

10. $\dfrac{6\sqrt[3]{81}}{\sqrt[3]{48}} = 6\sqrt[3]{\tfrac{27}{16}} = 18\sqrt[3]{\tfrac{1}{16}\times\tfrac{4}{4}} = 4\tfrac{1}{2}\sqrt[3]{4}.$

11. $\dfrac{\frac{1}{2}\sqrt{5}}{\frac{1}{3}\sqrt{2}} = \tfrac{3}{2}\sqrt{\tfrac{5}{2}} = \tfrac{3}{4}\sqrt{10}.$

12. $\dfrac{\frac{1}{2}\sqrt[4]{25}}{\frac{1}{3}\sqrt[4]{10}} = \tfrac{3}{2}\sqrt[4]{\tfrac{5}{2}} = \tfrac{3}{4}\sqrt[4]{40}.$

13. $\dfrac{\frac{2}{3}\sqrt{4\frac{1}{2}}+8\sqrt{12\frac{1}{2}}}{\sqrt{98}} = \dfrac{\sqrt{2}+20\sqrt{2}}{7\sqrt{2}} = \dfrac{21\sqrt{2}}{7\sqrt{2}} = 3.$

14. $\dfrac{12\sqrt[3]{\frac{2}{3}}+2\sqrt[3]{18}}{\sqrt[3]{72}} = 4\sqrt{\tfrac{18}{72}}+2\sqrt[3]{\tfrac{18}{72}} = 6\sqrt[3]{\tfrac{18}{72}} = 6\sqrt[3]{\tfrac{1}{4}} = 3\sqrt[3]{2}.$

15. $\dfrac{12\sqrt[3]{60}}{8\sqrt[3]{3\frac{3}{4}}+\sqrt[3]{240}} = \dfrac{12\sqrt[3]{60}}{4\sqrt[3]{30}+2\sqrt[3]{30}} = \dfrac{12\sqrt[3]{60}}{6\sqrt[3]{30}} = 2\sqrt[3]{2}.$

16. $\dfrac{2\sqrt{32}+4\sqrt{2}}{\sqrt{18}} = \dfrac{8\sqrt{2}+4\sqrt{2}}{3\sqrt{2}} = \dfrac{12\sqrt{2}}{3\sqrt{2}} = 4.$

17. $\dfrac{\sqrt{3}+6\sqrt{\frac{1}{3}}}{3\sqrt{2}} = \frac{1}{3}\sqrt{\frac{3}{2}}+2\sqrt{\frac{1}{6}} = \frac{1}{6}\sqrt{6}+\frac{1}{3}\sqrt{6} = (\frac{1}{6}+\frac{1}{3})\sqrt{6} = \frac{1}{2}\sqrt{6}.$

18. $\dfrac{\frac{1}{3}\sqrt{\frac{1}{2}}}{\sqrt{2}+3\sqrt{\frac{1}{2}}} = \dfrac{\frac{1}{6}\sqrt{2}}{\sqrt{2}+\frac{3}{2}\sqrt{2}} = \dfrac{\frac{1}{6}\sqrt{2}}{\frac{5}{2}\sqrt{2}} = \frac{1}{6}\times\frac{2}{5} = \frac{1}{15}.$

Art. 297.

1. $\dfrac{\sqrt[6]{72}}{\sqrt{2}} = \dfrac{72^{\frac{1}{6}}}{2^{\frac{1}{2}}} = \dfrac{72^{\frac{1}{6}}}{2^{\frac{3}{6}}} = \dfrac{\sqrt[6]{72}}{\sqrt[6]{8}} = \sqrt[6]{9} = \sqrt[3]{3}.$

2. $\dfrac{\sqrt{3}}{\sqrt[3]{6}} = \dfrac{3^{\frac{1}{2}}}{6^{\frac{1}{3}}} = \dfrac{3^{\frac{3}{6}}}{6^{\frac{2}{6}}} = \sqrt[6]{\frac{27}{36}} = \sqrt[6]{\frac{3}{4}} = \frac{1}{2}\sqrt[6]{48}.$

3. $\dfrac{\sqrt[6]{8a^2b^3}}{\sqrt[4]{2ab^2}} = \dfrac{(8a^2b^3)^{\frac{1}{6}}}{(2ab^2)^{\frac{1}{4}}} = \dfrac{(8a^2b^3)^{\frac{2}{12}}}{(2ab^2)^{\frac{3}{12}}} = \sqrt[12]{\dfrac{64a^4b^6}{8a^3b^6}} = \sqrt[12]{8a}.$

4. $\dfrac{\sqrt{30}}{\sqrt[3]{60}} = \dfrac{30^{\frac{1}{2}}}{60^{\frac{1}{3}}} = \dfrac{30^{\frac{3}{6}}}{60^{\frac{2}{6}}} = \sqrt[6]{\frac{27000}{3600}} = \frac{1}{2}\sqrt[6]{480}.$

5. $\dfrac{\sqrt{\frac{3}{2}}}{\sqrt[3]{\frac{2}{3}}} = \dfrac{(\frac{3}{2})^{\frac{1}{2}}}{(\frac{2}{3})^{\frac{1}{3}}} = \dfrac{(\frac{3}{2})^{\frac{3}{6}}}{(\frac{2}{3})^{\frac{2}{6}}} = \sqrt[6]{\frac{27}{8}\times\frac{9}{4}} = \sqrt[6]{\frac{243}{32}} = \frac{1}{2}\sqrt[6]{486}.$

6. $\dfrac{1\frac{1}{3}\sqrt[3]{2\frac{2}{3}}}{\frac{2}{3}\sqrt{1\frac{1}{3}}} = \dfrac{\frac{4}{3}(\frac{8}{3})^{\frac{2}{6}}}{\frac{2}{3}(\frac{4}{3})^{\frac{3}{6}}} = \dfrac{\frac{4}{3}\sqrt[6]{\frac{64}{9}}}{\frac{2}{3}\sqrt[6]{\frac{64}{27}}} = 2\sqrt[6]{3}.$

7. $\dfrac{\sqrt[6]{64a^5b^8}}{\sqrt[3]{4a^2b}} = \dfrac{(64a^5b^8)^{\frac{1}{6}}}{(4a^2b)^{\frac{1}{3}}} = \sqrt[6]{\dfrac{64a^5b^8}{16a^4b^2}} = \sqrt[6]{4ab}$

8. $\dfrac{\sqrt{8}}{\sqrt[4]{12}} = \dfrac{(8)^{\frac{1}{2}}}{(12)^{\frac{1}{4}}} = \sqrt[4]{\frac{64}{12}} = \sqrt[4]{\frac{16}{3}} = \frac{2}{3}\sqrt[4]{27}.$

9. $\dfrac{\sqrt{\frac{2}{3}}}{\sqrt[3]{\frac{1}{3}}} = \dfrac{(\frac{2}{3})^{\frac{1}{2}}}{(\frac{1}{3})^{\frac{1}{3}}} = \dfrac{(\frac{2}{3})^{\frac{3}{6}}}{(\frac{1}{3})^{\frac{2}{6}}} = \sqrt[6]{\dfrac{\frac{8}{27}}{\frac{1}{9}}} = \sqrt[6]{\frac{8}{3}} = \frac{1}{3}\sqrt[6]{1944}.$

10. $\dfrac{\sqrt[6]{12a^2x^4}}{\sqrt[4]{2ax^2}} = \dfrac{(12a^2x^4)^{\frac16}}{(2ax^2)^{\frac14}} = \sqrt[12]{\dfrac{144a^4x^8}{8a^3x^6}} = \sqrt[12]{18ax^2}.$

11. $\dfrac{\sqrt[3]{2a^2x^3}}{\sqrt[5]{2a^2x^3}} = \dfrac{(2a^2x^3)^{\frac13}}{(2a^2x^3)^{\frac15}} = \sqrt[15]{\dfrac{(2a^2x^3)^5}{(2a^2x^3)^3}} = \sqrt[15]{4a^4x^6}.$

12. $\dfrac{125\sqrt{x^3y^3}}{25\sqrt[3]{x^2y^2}} = 5\,\dfrac{(x^3y^3)^{\frac12}}{(x^2y^2)^{\frac13}} = 5\sqrt[6]{\dfrac{x^9y^9}{x^4y^4}} = 5\sqrt[6]{x^5y^5}.$

13. $\dfrac{6\sqrt[5]{8}}{2\sqrt[3]{2}} = 3\,\dfrac{(8)^{\frac15}}{(2)^{\frac13}} = 3\sqrt[15]{\tfrac{512}{32}} = 3\sqrt[15]{16}.$

14. $\dfrac{\sqrt[5]{18}}{\sqrt{2}} = \dfrac{18^{\frac15}}{2^{\frac12}} = \dfrac{18^{\frac{2}{10}}}{2^{\frac{5}{10}}} = \sqrt[10]{\tfrac{324}{32}} = \tfrac12\sqrt[10]{10368}.$

15. $\dfrac{\sqrt[3]{9a^2b^4}}{\sqrt[4]{27a^2b^5}} = \dfrac{(9a^2b^4)^{\frac{4}{12}}}{(27a^2b^5)^{\frac{3}{12}}} = \sqrt[12]{\dfrac{9^4a^8b^{16}}{27^3a^6b^{15}}} = \sqrt[12]{\tfrac13 a^2b} = \tfrac13\sqrt[12]{3^{11}a^2b}.$

16. $\dfrac{5\sqrt[6]{500}}{\sqrt{5}} = \dfrac{5\,(500)^{\frac16}}{(5)^{\frac12}} = 5\sqrt[6]{\tfrac{500}{125}} = 5\sqrt[6]{4} = 5\sqrt[3]{2}.$

17. $\dfrac{6\sqrt[6]{128}}{\sqrt[4]{4}} = \dfrac{6\,(128)^{\frac16}}{(2)^{\frac12}} = 6\sqrt[6]{\tfrac{128}{8}} = 6\sqrt[6]{16} = 6\sqrt[3]{4}.$

18. $\dfrac{3\sqrt[6]{392}}{4\sqrt[10]{32}} = \dfrac{3\sqrt[6]{392}}{4\sqrt{2}} = \dfrac{3\,(392)^{\frac16}}{4\,(8)^{\frac16}} = \tfrac34\sqrt[6]{\tfrac{392}{8}} = \tfrac34\sqrt[3]{7}.$

19. $\dfrac{\sqrt[3]{a^2}-\sqrt[5]{b^4}}{\sqrt[3]{a}-\sqrt[5]{b^2}} = \dfrac{(\sqrt[3]{a}-\sqrt[5]{b^2})(\sqrt[3]{a}+\sqrt[5]{b^2})}{\sqrt[3]{a}-\sqrt[5]{b^2}} = \sqrt[3]{a}+\sqrt[5]{b^2}.$

20. $\dfrac{\sqrt[3]{x^2}+2\sqrt[12]{x^4y^3}+\sqrt{y}}{\sqrt[3]{x}+\sqrt[4]{y}} = \dfrac{(\sqrt[3]{x}+\sqrt[4]{y})^2}{\sqrt[3]{x}+\sqrt[4]{y}} = \sqrt[3]{x}+\sqrt[4]{y}.$

21. $\dfrac{\sqrt[3]{a^2}+\sqrt[6]{a^2b^3}+b}{\sqrt[3]{a}-\sqrt[12]{a^2b^3}+\sqrt{b}} = \dfrac{(\sqrt[3]{a}-\sqrt[12]{a^2b^3}+\sqrt{b})(\sqrt[3]{a}+\sqrt[12]{a^2b^3}+\sqrt{b})}{\sqrt[3]{a}-\sqrt[12]{a^2b^3}+\sqrt{b}}$

$= \sqrt[3]{a}+\sqrt[12]{a^2b^3}+\sqrt{b}.$

Art. 299.

1. Given $\sqrt{x+9}=4$; to find the value of x.

Squaring, $x+9=16$.

Whence, $x=\ \ 7$.

2. Given $\sqrt{4x+1}=3$; to find the value of x.

Squaring, $4x+1=9$.

Whence, $x=2$.

3. Given $\sqrt{x-9}=2$; to find the value of x.

Squaring, $x-9=\ \ 4$.

Whence, $x=13$.

4. Given $\sqrt{x^2-144}=x-4$; to find the value of x.

Squaring, $x^2-144=x^2-8x+16$.

Transposing and collecting terms,

$$8x=160.$$

Whence, $x=20$.

5. Given $\sqrt{x^2+5x+3}=x+2$; to find the value of x.

Squaring, $x^2+5x+3=x^2+4x+4$.

Transposing and collecting terms,

$$x=1.$$

6. Given $\sqrt{x+9}=2\sqrt{x}-3$; to find the value of x.

Squaring, $x+9=4x-12\sqrt{x}+9$.

Transposing, $3x=12\sqrt{x}$.

Squaring, $9x^2=144x$.

Whence, $x=16$.

7. Given $\sqrt{x+16} = \sqrt{x}+2$; to find the value of x.

Squaring, $x+16 = x+4\sqrt{x}+4.$

Transposing, $4\sqrt{x} = 12.$

Squaring, $16x = 144.$

Whence, $x = 9.$

8. Given $\sqrt{x-16} = 8-\sqrt{x}$; to find the value of x.

Squaring, $x-16 = 64-16\sqrt{x}+x.$

Transposing, $16\sqrt{x} = 80.$

Whence, $\sqrt{x} = 5,$

and $x = 25.$

9. Given $\sqrt{x-32}+\sqrt{x} = 16$; to find the value of x.

Transposing, $\sqrt{x-32} = 16-\sqrt{x}.$

Squaring, $x-32 = 256-32\sqrt{x}+x.$

Transposing, $32\sqrt{x} = 288.$

Whence, $\sqrt{x} = 9,$

and $x = 81.$

10. Given $\sqrt{x} = \sqrt{2}+\sqrt{x-2}$; to find the value of x.

Transposing, $\sqrt{x-2} = \sqrt{x}-\sqrt{2}.$

Squaring, $x-2 = x-2\sqrt{2x}+2.$

Transposing, $2\sqrt{2x} = 4.$

Whence, $\sqrt{2x} = 2,$

and $x = 2.$

11. Given $\sqrt{x^2-7} = 7-x$; to find the value of x.

Squaring, $x^2-7 = 49-14x+x^2.$

Transposing, $14x = 56.$

Whence, $x = 4.$

12. Given $\sqrt{4x+9} = 2\sqrt{x}+1$; to find the value of x.

Squaring, $4x+9 = 4x+4\sqrt{x}+1$.

Transposing, $4\sqrt{x} = 8$.

Whence, $\sqrt{x} = 2$,

and $x = 4$.

13. Given $\frac{1}{\sqrt{x}} - \sqrt{x} = \sqrt{x+1}$; to find the value of x.

Clearing of fractions, $1-x = \sqrt{x^2+x}$.

Squaring, $1-2x+x^2 = x^2+x$.

Transposing, $3x = 1$.

Whence, $x = \frac{1}{3}$.

14. Given $\sqrt{5+\sqrt{x-4}} = 3$; to find the value of x.

Squaring, $5+\sqrt{x-4} = 9$.

Transposing, $\sqrt{x-4} = 4$.

Squaring, $x-4 = 16$.

Whence, $x = 20$.

15. Given $\sqrt{4x^2+2x-16} = 2x-2$; to find the value of x.

Squaring, $4x^2+2x-16 = 4x^2-8x+4$.

Transposing, $10x = 20$.

Whence, $x = 2$.

16. Given $\sqrt{x-5}-7 = \sqrt{x-12}$; to find the value of x.

Squaring, $x-5-14\sqrt{x-5}+49 = x-12$.

Transposing, $14\sqrt{x-5} = 56$.

Whence, $\sqrt{x-5} = 4$.

Squaring and transposing, $x = 21$.

17. Given $\sqrt{x-5}+\sqrt{x+7}=6$; to find the value of x.

Transposing, $\sqrt{x-5}=6-\sqrt{x+7}$.

Squaring, $x-5=36-12\sqrt{x+7}+x+7$.

Transposing, $12\sqrt{x+7}=48$,

and $\sqrt{x+7}=4$.

Squaring and transposing, $x=9$.

18. Given $\sqrt{5+x}+\sqrt{x}=\dfrac{15}{\sqrt{5+x}}$; to find the value of x.

Clearing of fractions, $5+x+\sqrt{5x+x^2}=15$.

Transposing, $\sqrt{5x+x^2}=10-x$.

Squaring, $5x+x^2=100-20x+x^2$.

Transposing, $25x=100$.

Whence, $x=4$.

19. Given $\sqrt{2x-1}+\sqrt{2x+4}=5$; to find the value of x.

Transposing, $\sqrt{2x-1}=5-\sqrt{2x+4}$.

Squaring, $2x-1=25-10\sqrt{2x+4}+2x+4$.

Transposing, $10\sqrt{2x+4}=30$.

Whence, $\sqrt{2x+4}=3$.

Squaring, $2x+4=9$,

and $x=2\frac{1}{2}$.

20. Given $(\sqrt{x}-2)^2+4\sqrt{x}=13$; to find the value of x.

Transposing, $(\sqrt{x}-2)^2=13-4\sqrt{x}$,

and $x-4\sqrt{x}+4=13-4\sqrt{x}$.

Whence, $x=9$.

21. Given $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$; to find the value of x.

Clearing of fractions, $\sqrt{x^2-9x}+x-9=36.$

Transposing, $\sqrt{x^2-9x}=45-x.$

Squaring, $x^2-9x=2025-90x+x^2.$

Transposing, $81x=2025.$

Whence, $x=25.$

22. Given $\sqrt{5x+10}-\sqrt{5x}=2$; to find the value of x.

Transposing, $\sqrt{5x+10}=2+\sqrt{5x}.$

Squaring, $5x+10=4+4\sqrt{5x}+5x.$

Transposing, $4\sqrt{5x}=6.$

Squaring, $80x=36.$

Whence, $x=\frac{9}{20}.$

23. Given $\sqrt{13+\sqrt{7+\sqrt{3+\sqrt{x}}}}=4$; to find the value of x.

Squaring, $13+\sqrt{7+\sqrt{3+\sqrt{x}}}=16.$

Transposing, $\sqrt{7+\sqrt{3+\sqrt{x}}}=3.$

Squaring and combining, $\sqrt{3+\sqrt{x}}=2.$

Squaring and combining, $\sqrt{x}=1.$

Squaring, $x=1.$

24. Given $\sqrt{21+\sqrt{7+\sqrt{x-4}}}=5$; to find the value of x.

Squaring and combining, $\sqrt{7+\sqrt{x-4}}=4.$

Squaring and combining, $\sqrt{x-4}=9.$

Squaring and combining, $x=85.$

Art. 301.

1. Given $\dfrac{\sqrt{x}+3}{\sqrt{x}-1} = \dfrac{\sqrt{x}+13}{\sqrt{x}+1}$.

By (4), $$\frac{4}{\sqrt{x}-1} = \frac{12}{\sqrt{x}+1}.$$

Reducing, $$\frac{1}{\sqrt{x}-1} = \frac{3}{\sqrt{x}+1}.$$

Clearing of fractions, $$\sqrt{x}+1 = 3\sqrt{x}-3.$$

Whence, $$x = 4.$$

2. Given $\dfrac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}} = 4$.

By (7), $$\frac{2\sqrt{3x+1}}{2\sqrt{3x}} = \frac{5}{3}.$$

Clearing of fractions, $$3\sqrt{3x+1} = 5\sqrt{3x}.$$

Squaring, $$27x+9 = 75x.$$

Whence, $$x = \tfrac{3}{16}.$$

3. Given $\dfrac{3\sqrt{x-4}}{\sqrt{x+2}} = \dfrac{3\sqrt{x+15}}{\sqrt{x+40}}$.

Reducing and squaring, $$\frac{x-4}{x+2} = \frac{x+15}{x+40}.$$

By (4), $$\frac{-6}{x+2} = \frac{-25}{x+40}.$$

Whence, $$x = 10.$$

4. Given $\dfrac{1+x+\sqrt{2x+x^2}}{1+x-\sqrt{2x+x^2}} = \dfrac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}-\sqrt{x}}$.

By (7), $$\frac{2+2x}{2\sqrt{2x+x^2}} = \frac{2\sqrt{x+2}}{2\sqrt{x}}.$$

Reducing and squaring, $\dfrac{(1+x)^2}{2x+x^2}=\dfrac{x+2}{x}.$

By (4), $\dfrac{1}{2x+x^2}=\dfrac{2}{x}.$

Whence, $x=-1\frac{1}{2}.$

5. Given $\dfrac{\sqrt{x-2}}{\sqrt{x+3}}=\dfrac{\sqrt{x+1}}{\sqrt{x+21}}.$

Squaring, $\dfrac{x-2}{x+3}=\dfrac{x+1}{x+21}.$

By (4), $\dfrac{-5}{x+3}=\dfrac{-20}{x+21}.$

Whence, $x=3.$

6. Given $\dfrac{\sqrt{2x-7}+\sqrt{x}}{\sqrt{2x-7}-\sqrt{x}}=\dfrac{19}{3}.$

By (7), $\dfrac{\sqrt{2x-7}}{\sqrt{x}}=\dfrac{22}{16}=\dfrac{11}{8}.$

Squaring, $\dfrac{2x-7}{x}=\dfrac{121}{64}.$

Whence, $x=64.$

7. Given $\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}}=\dfrac{n}{1}$

By (7), $\dfrac{1}{-\sqrt{1-x}}=\dfrac{n+1}{n-1}.$

Squaring, $\dfrac{1}{x-1}=\dfrac{n^2+2n+1}{n^2-2n+1}.$

By (5), $\dfrac{x}{1}=\dfrac{4n}{(n+1)^2}.$

Whence. $x=\dfrac{4n}{(n+1)^2}.$

8. Given $\dfrac{\sqrt{12x+1}+\sqrt{12x}}{\sqrt{12x+1}-\sqrt{12x}}=\dfrac{18}{1}$.

By (7), $$\frac{\sqrt{12x+1}}{\sqrt{12x}}=\frac{19}{17}.$$

Squaring, etc., $$x=\tfrac{289}{864}.$$

9. Given $\dfrac{\sqrt{4x+1}+2\sqrt{x}}{\sqrt{4x+1}-2\sqrt{x}}=\dfrac{9}{1}$

By (7), $$\frac{2\sqrt{4x+1}}{4\sqrt{x}}=\frac{10}{8},$$

or, $$\frac{\sqrt{4x+1}}{2\sqrt{x}}=\frac{5}{4}.$$

Squaring, etc., $$x=\tfrac{4}{9}.$$

10. Given $\dfrac{\sqrt{x-1}}{\sqrt{x+1}}=\dfrac{2(x-1)}{x+1}$.

Squaring, $$\frac{x-1}{x+1}=\frac{4(x-1)^2}{(x+1)^2}.$$

Dividing through by $\dfrac{x-1}{x+1}$, $$1=\frac{4(x-1)}{x+1}.$$

Whence, $$x=1\tfrac{2}{3}.$$

11. Given $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$.

By (7), $$\frac{2\sqrt{6x}}{-4}=\frac{8\sqrt{6x}-3}{-15}.$$

Clearing of fractions, etc., $$2\sqrt{6x}=12,$$

Whence, $$x=6.$$

12. Given $\dfrac{\sqrt{9x}-4}{\sqrt{x}+2}=\dfrac{\sqrt{9x}+15}{\sqrt{x}+40}$

Simplifying, $$\frac{3\sqrt{x}-4}{\sqrt{x}+2}=\frac{3\sqrt{x}+15}{\sqrt{x}+40}.$$

Multiplying denominator by 3, $\frac{3\sqrt{x}-4}{3\sqrt{x}+6} = \frac{3\sqrt{x}+15}{3\sqrt{x}+120}$.

By (7), and reducing, $3\sqrt{x}+1 = \frac{6\sqrt{x}+135}{21}$.

Whence, $x = 4$.

13. Given $\frac{\sqrt{x}+2a}{\sqrt{x}+b} = \frac{\sqrt{x}+4a}{\sqrt{x}+3b}$.

By (7), $\frac{2\sqrt{x}+2a+b}{2a-b} = \frac{2\sqrt{x}+4a+3b}{4a-3b}$.

Clearing of fractions,

$$8a\sqrt{x}-6b\sqrt{x}+8a^2-2ab-3b^2 = 4a\sqrt{x}-2b\sqrt{x}+8a^2+2ab-3b^2.$$

Transposing and uniting,

$$(4a-4b)\sqrt{x} = 4ab.$$

Whence, $x = \left(\frac{ab}{a-b}\right)^2$.

14. Given $\frac{\sqrt{ax}-b}{\sqrt{ax}+b} = \frac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$.

By (7), and reducing, $\sqrt{ax} = \frac{6\sqrt{ax}+3b}{7}$.

Whence, $x = \frac{9b^2}{a}$.

15. Given $\frac{\sqrt{x}+\sqrt{b}}{\sqrt{x}-\sqrt{b}} = \frac{a}{b}$.

By (7), and reducing, $\frac{\sqrt{x}}{\sqrt{b}} = \frac{a+b}{a-b}$.

Squaring, $\frac{x}{b} = \frac{(a+b)^2}{(a-b)^2}$.

Whence, $x = \frac{b(a+b)^2}{(a-b)^2}$.

Art. 310.

18. $$\frac{3x^2}{4}-\frac{15x^2+8}{6}=2x^2-3.$$

Clearing of fractions, $9x^2-30x^2-16=24x^2-36.$

Whence, $x=\pm\frac{2}{3}.$

19. $$\frac{x^2}{5}-\frac{x^2-10}{15}=7-\frac{50+x^2}{25}.$$

Clearing of fractions, $15x^2-5x^2+50=525-150-3x^2.$

Whence, $x=\pm 5.$

20. $$\frac{3x^2-27}{x^2+3}+\frac{90+4x^2}{x^2+9}=7.$$

Clearing of fractions,

$$3x^4-243+270+102x^2+4x^4=7x^4+84x^2+189.$$

Whence, $x=\pm 3.$

21. $$\frac{4x^2+5}{10}-\frac{2x^2-5}{15}=\frac{7x^2-25}{20}.$$

Clearing of fractions,

$$24x^2+30-8x^2+20=21x^2-75.$$

Whence, $x=\pm 5.$

22. $$\frac{10x^2+17}{18}-\frac{12x^2+2}{11x^2-8}=\frac{5x^2-4}{9}.$$

Clearing of fractions,

$$110x^4+107x^2-136-216x^2-36=110x^4-168x^2+64.$$

Transposing and uniting, $59x^2=236,$

or, $x^2=4.$

Extracting square root, $x=\pm 2.$

23. $$\frac{14x^2+16}{21}-\frac{2x^2+8}{8x^2-11}=\frac{2x^2}{3}.$$

Clearing of fractions,

$$112x^4-26x^2-176-42x^2-108=112x^4-154x^2.$$

Transposing and uniting, $86x^2=344,$

or, $x^2=4.$

Extracting square root, $x=\pm 2.$

24.
$$\frac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}}=\frac{4}{1}.$$

By (7), and reducing, $\qquad \dfrac{2x}{\sqrt{4x^2-1}}=\dfrac{5}{3}.$

Squaring, $\qquad \dfrac{4x^2}{4x^2-1}=\dfrac{25}{9}.$

Whence, $\qquad x=\pm\frac{5}{8}.$

25.
$$\frac{2}{x+\sqrt{2-x^2}}+\frac{2}{x-\sqrt{2-x^2}}=x.$$

Adding terms of first member, and reducing,

$$\frac{2}{x^2-1}=1.$$

Whence, $\qquad x=\pm\sqrt{3}.$

26.
$$\frac{\sqrt{1+x}}{1+\sqrt{1+x}}=\frac{\sqrt{1-x}}{1-\sqrt{1-x}}.$$

Clearing of fractions,

$$\sqrt{1+x}-\sqrt{1-x^2}=\sqrt{1-x}+\sqrt{1-x^2}.$$

Transposing and uniting, $\qquad 2\sqrt{1-x^2}=\sqrt{1+x}+\sqrt{1-x}.$

Squaring, $\qquad 4-4x^2=2+2\sqrt{1-x^2}.$

Transposing, uniting, and dividing by 2,

$$1-2x^2=\sqrt{1-x^2}.$$

Squaring, etc., $\qquad x=\pm\frac{1}{2}\sqrt{3}.$

27.
$$\frac{1}{\sqrt{1-x}+1}+\frac{1}{\sqrt{1+x}-1}=\frac{1}{x}.$$

Rationalize the first two denominators:

$$\frac{1}{\sqrt{1-x}+1}\times\frac{\sqrt{1-x}-1}{\sqrt{1-x}-1}=\frac{(\sqrt{1-x}-1)}{1-x-1};$$

$\frac{1}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \frac{\sqrt{1+x}+1}{1+x-1}.$ $\frac{\sqrt{1-x}-1}{-x} + \frac{\sqrt{1+x}+1}{x} = \frac{1}{x}.$

$-\sqrt{1-x}+1 + \sqrt{1+x}+1 = 1.$ $\sqrt{1+x} - \sqrt{1-x} = -1.$

$1+x-2\sqrt{1-x^2}+1-x = 1.$ $-2\sqrt{1-x^2} = -1.$ $4-4x^2 = 1.$ $4x^2 = 3.$

$x^2 = \frac{3}{4}.$ $x = \pm\frac{1}{2}\sqrt{3}.$

28. $$\frac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}} = \frac{x}{2a}.$$

By (7), and reducing, $$\frac{\sqrt{x+2a}}{-\sqrt{x-2a}} = \frac{x+2a}{x-2a}.$$

Squaring, $$\frac{x+2a}{x-2a} = \frac{x^2+4ax+4a^2}{x^2-4ax+4a^2}.$$

By (7), and reducing, $$\frac{x}{2a} = \frac{x^2+4a^2}{4ax}.$$

Whence, $$x = \pm 2a.$$

29. $$\frac{\sqrt{3x^2+4}+2}{\sqrt{3x^2+4}-2} = \frac{3}{1}.$$

By (7), and reducing, $$\frac{\sqrt{3x^2+4}}{2} = 2.$$

Whence, $$x = \pm 2.$$

30. $$\frac{1}{\sqrt{1-x^2}+1} + \frac{1}{\sqrt{1-x^2}-1} = \frac{1}{x^2}.$$

Adding terms of first member, and reducing,

$2\sqrt{1-x^2} = -1.$ Whence, $x = \pm\frac{1}{2}\sqrt{3}.$

31. $$\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \frac{1}{2}.$$

By (7), and reducing, $$\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}} = -3.$$

Whence, $$x = \pm\frac{1}{2}\sqrt{5}.$$

32. $$x\sqrt{6+x^2} = 1+x^2.$$

Squaring, $$6x^2+x^4 = 1+2x^2+x^4.$$

Whence, $$x = \pm \tfrac{1}{2}.$$

33. $$\frac{1}{\sqrt{a-x}+\sqrt{a}} + \frac{1}{\sqrt{a+x}-\sqrt{a}} = \frac{\sqrt{a}}{x}.$$

Multiplying numerators and denominators by same quantity,

$$\frac{1}{\sqrt{a-x}+\sqrt{a}} \times \frac{\sqrt{a-x}-\sqrt{a}}{\sqrt{a-x}-\sqrt{a}} = \frac{\sqrt{a-x}-\sqrt{a}}{-x}.$$

and $$\frac{1}{\sqrt{a+x}-\sqrt{a}} \times \frac{\sqrt{a+x}+\sqrt{a}}{\sqrt{a+x}+\sqrt{a}} = \frac{\sqrt{a+x}+\sqrt{a}}{x}.$$

Whence,

$$-\frac{\sqrt{a-x}-\sqrt{a}}{x} + \frac{\sqrt{a+x}+\sqrt{a}}{x} = \frac{\sqrt{a}}{x},$$

or, $$\sqrt{a+x} = \sqrt{a-x}-\sqrt{a}.$$

Squaring, $$a+x = a-x+a-2\sqrt{a(a-x)}.$$

Whence, $$2\sqrt{a(a-x)} = a-2x.$$

Squaring, $$4a^2-4ax = a^2-4ax+4x^2.$$

Whence, $$x = \pm\frac{1}{2}a\sqrt{3}.$$

34. $$\frac{ax+1+\sqrt{a^2x^2-1}}{ax+1-\sqrt{a^2x^2-1}} = \frac{bx}{2}.$$

By (7), and reducing, $$\frac{ax+2}{\sqrt{a^2x^2-1}} = \frac{bx+2}{bx-2}.$$

Squaring, $$\frac{a^2x^2+2ax+1}{a^2x^2-1} = \frac{b^2x^2+4bx+4}{b^2x^2-4bx+4}.$$

By (7), and reducing, $$ax = \frac{b^2x^2+4}{4bx}.$$

Whence, $$x = \pm\frac{2}{\sqrt{4ab-b^2}}.$$

35. $$\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}=\frac{x}{b}.$$

Clearing of fractions, $$ab+b\sqrt{a^2-x^2}=x^2.$$

Transposing, $$b\sqrt{a^2-x^2}=x^2-ab.$$

Squaring, $$a^2b^2-b^2x^2=x^4-2abx^2+a^2b^2.$$

Transposing and uniting, $$x^4=2abx^2-b^2x^2.$$

Dividing by x^2, $$x^2=2ab-b^2.$$

Extracting square root, $$x=\pm\sqrt{2ab-b^2}.$$

36. $$\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}=\frac{b}{1}.$$

By (7), and reducing, $$\frac{\sqrt{a^2+x^2}}{-a}=\frac{b+1}{b-1}.$$

Squaring, $$\frac{a^2+x^2}{a^2}=\frac{b^2+2b+1}{b^2-2b+1}.$$

By (7), and reducing, $$\frac{2a^2+x^2}{x^2}=\frac{b^2+1}{2b}.$$

Clearing of fractions, $$4a^2b+2bx^2=b^2x^2+x^2.$$

Whence, $$x^2=\frac{4a^2b}{1-2b+b^2}.$$

Extracting square root, $$x=\pm\frac{2a\sqrt{b}}{1-b}.$$

Art. 316.

31. $$\frac{4x}{x+7}-\frac{x-7}{2x+3}-2=0.$$

Clearing of fractions and uniting,

$$3x^2-22x=-7.$$

Multiplying by 3, $$9x^2-66x=-21.$$

Adding 121, $$9x^2-66x+121=100.$$

Extracting square root, $$3x-11=\pm 10.$$

Whence, $$x=7 \text{ or } \tfrac{1}{3}.$$

32. $$\frac{48}{x+3}+5=\frac{165}{x+10}.$$

Clearing of fractions and uniting,

$$5x^2-52x=-135.$$

Multiplying by 5, $$25x^2-260x=-675.$$

Adding 676, $$25x^2-260x+676=1.$$

Extracting square root, $$5x-26=\pm 1.$$

Whence, $$x=5\tfrac{2}{5} \text{ or } 5.$$

33. $$\frac{8x}{x+2}-\frac{20}{3x}-6=0.$$

Clearing of fractions and uniting,

$$6x^2-56x=40.$$

Multiplying by 6, $$36x^2-336x=240.$$

Adding 784, $$36x^2-336x+784=1024.$$

Extracting square root, $$6x-28=\pm 32.$$

Whence, $$x=10 \text{ or } -\tfrac{2}{3}.$$

Art. 318.

1.
$$x-5\sqrt{x} = 14.$$
$$-5\sqrt{x} = 14-x.$$
Squaring, $25x = 196-28x+x^2.$

Transposing and uniting, $x^2-53x = -196.$

Adding $(\frac{53}{2})^2$, $x^2-3x+\frac{2809}{4} = \frac{2025}{4}.$

Extracting square root, $x-\frac{53}{2} = \pm\frac{45}{2}.$

Whence, $x = \frac{53}{2} \pm \frac{45}{2} = 49$ or $4.$

2.
$$x+\sqrt{x+5} = 7.$$
$$\sqrt{x+5} = 7-x.$$
Squaring, $x+5 = 49-14x+x^2.$

Transposing and uniting, $x^2-15x = -44.$

Adding $(\frac{15}{2})^2$, $x^2-15x+\frac{225}{4} = \frac{49}{4}.$

Extracting square root, $x-\frac{15}{2} = \pm\frac{7}{2}.$

Whence, $x = \frac{15}{2} \pm \frac{7}{2} = 11$ or $4.$

3.
$$\sqrt{4x+17}+\sqrt{x+1} = 4.$$
$$\sqrt{4x+17} = 4-\sqrt{x+1}.$$
Squaring, $4x+17 = 16-8\sqrt{x+1}+x+1.$

Transposing and uniting, $8\sqrt{x+1} = -3x.$

Squaring, $64x+64 = 9x^2.$

Transposing, $9x^2-64x = 64.$

Adding $(\frac{32}{3})^2$, $9x^2-64x+\frac{1024}{9} = \frac{1600}{9}.$

Extracting square root, $3x-\frac{32}{3} = \pm\frac{40}{3}.$

Whence, $3x = \frac{32}{3} \pm \frac{40}{3} = 8$ or $-\frac{8}{3}.$

4. $$\sqrt{x^2-4x-9} = 6.$$

Squaring, $$x^2-4x-9 = 36.$$

Transposing and uniting, $$x^2-4x = 45.$$

Adding 4, $$x^2-4x+4 = 49.$$

Extracting square root, $$x-2 = \pm 7.$$

Whence, $$x = 2 \pm 7 = 9 \text{ or } -5.$$

5. $$\sqrt{3x^2+7x-2} = 2.$$

Squaring, $$3x^2+7x-2 = 4.$$

Transposing and multiplying by 3,

$$9x^2+21x = 18.$$

Adding $(\frac{7}{2})^2$, $$9x^2+21x+\tfrac{49}{4} = \tfrac{121}{4}.$$

Extracting square root, $$3x+\tfrac{7}{2} = \pm\tfrac{11}{2}.$$

Whence, $$3x = -\tfrac{7}{2} \pm \tfrac{11}{2},$$

and $$x = \tfrac{2}{3} \text{ or } -3.$$

6. $$\sqrt{5x+10} = 8-x.$$

Squaring, $$5x+10 = 64-16x+x^2.$$

Transposing and uniting, $$x^2-21x = -54.$$

Adding $(\frac{21}{2})^2$, $$x^2-21x+\tfrac{441}{4} = \tfrac{225}{4}.$$

Extracting square root, $$x-\tfrac{21}{2} = \pm\tfrac{15}{2}.$$

Whence, $$x = \tfrac{21}{2} \pm \tfrac{15}{2} = 18 \text{ or } 3.$$

7. $$x+2\sqrt{x^2+x+5} = 14.$$

$$2\sqrt{x^2+x+5} = 14-x.$$

Squaring, $$4x^2+4x+20 = 196-28x+x^2.$$

Transposing and uniting, $$3x^2+32x = 176.$$

Multiplying by 3, $$9x^2+96x = 528.$$

Adding 256, $$9x^2+96x+256 = 784.$$

Extracting square root, $$3x+16 = \pm 28.$$

Whence, $$3x = -16 \pm 28,$$

and $$x = 4 \text{ or } -14\tfrac{2}{3}.$$

8. $$\frac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\frac{5}{11}.$$

By (7), and reducing, $$\frac{x}{\sqrt{x+1}}=\frac{8}{3}.$$

Squaring, $$\frac{x^2}{x+1}=\frac{64}{9}.$$

Clearing of fractions, $$9x^2=64x+64.$$

Transposing and adding $(\frac{32}{3})^2$,
$$9x^2-64x+\tfrac{1024}{9}=\tfrac{1600}{9}.$$

Extracting square root, $$3x-\tfrac{32}{3}=\pm\tfrac{40}{3}.$$

Whence, $$x=8 \text{ or } -\tfrac{8}{9}.$$

9. $$\frac{2(\sqrt{x}+1)}{\sqrt{x}+4}=\frac{4-\sqrt{x}}{\sqrt{x}}.$$

Clearing of fractions, $$2x+2\sqrt{x}=-x+16.$$

Transposing and uniting, $$2\sqrt{x}=16-3x.$$

Squaring, $$4x=256-96x+9x^2.$$

Transposing and uniting, $$9x^2-100x=-256.$$

Adding $(\frac{50}{3})^2$, $$9x^2-100x+\tfrac{2500}{9}=\tfrac{196}{9}.$$

Extracting square root, $$3x-\tfrac{50}{3}=\pm\tfrac{14}{3}.$$

Whence, $$x=4 \text{ or } 7\tfrac{1}{9}.$$

10. $$\sqrt{10x+6}=9-x.$$

Squaring, $$10x+6=81-18x+x^2.$$

Transposing and uniting, $$x^2-28x=-75.$$

Adding 14^2, $$x^2-28x+196=121.$$

Extracting square root, $$x-14=\pm 11.$$

Whence, $$x=14\pm 11=25 \text{ or } 3.$$

11.
$$\sqrt{6x+1}+\sqrt{x+4+\sqrt{6x+1}} = 2.$$

$$\sqrt{x+4+\sqrt{6x+1}} = 2-\sqrt{6x+1}.$$

Squaring, $$x+4+\sqrt{6x+1} = 4-4\sqrt{6x+1}+6x+1.$$

Transposing and uniting, $$5\sqrt{6x+1} = 5x+1.$$

Squaring, $$150x+25 = 25x^2+10x+1.$$

Transposing and uniting, $$25x^2-140x = 24.$$

Adding 14^2, $$25x^2-140x+196 = 220.$$

Extracting square root, $$5x-14 = \pm 2\sqrt{55}.$$

Whence, $$x = \tfrac{2}{5}(7 \pm \sqrt{55}).$$

12.
$$\sqrt{x+5} = \frac{12}{\sqrt{x+12}}.$$

Squaring, $$x+5 = \frac{144}{x+12}.$$

Clearing of fractions, $$x^2+17x+60 = 144.$$

Transposing and uniting, $$x^2+17x = 84.$$

Adding $(\frac{17}{2})^2$, $$x^2+17x+\tfrac{289}{4} = \tfrac{625}{4}.$$

Extracting square root, $$x+\tfrac{17}{2} = \pm \tfrac{25}{2}.$$

Whence, $$x = -\tfrac{17}{2} \pm \tfrac{25}{2} = 4 \text{ or } -21.$$

13.
$$\frac{5(3x-1)}{5\sqrt{x}+1} + \frac{2}{\sqrt{x}} = 3\sqrt{x}.$$

Clearing of fractions,

$$15\sqrt{x^3}-5\sqrt{x}+10\sqrt{x}+2 = 15\sqrt{x^3}+3x.$$

Transposing and uniting, $$5\sqrt{x} = 3x-2.$$

Squaring, $$25x = 9x^2-12x+4.$$

Transposing and uniting, $$9x^2-37x = -4.$$

Adding $(\frac{37}{6})^2$, $$9x^2-37x+\tfrac{1369}{36} = \tfrac{1225}{36}.$$

Extracting square root, $$3x-\tfrac{37}{6} = \pm\tfrac{35}{6}.$$

Whence, $$x = 4 \text{ or } \tfrac{1}{9}.$$

14. $$\sqrt{2x+1}+2\sqrt{x} = \frac{21}{\sqrt{2x+1}}.$$

Clearing of fractions,

$$2x+1+2\sqrt{2x^2+x} = 21.$$

Transposing and reducing, $$\sqrt{2x^2+x} = 10-x.$$

Squaring, $$2x^2+x = 100-20x+x^2.$$

Transposing and uniting, $$x^2+21x = 100.$$

Adding $(\frac{21}{2})^2$, $$x^2+21x+\tfrac{441}{4} = \tfrac{841}{4}.$$

Extracting square root, $$x+\tfrac{21}{2} = \pm\tfrac{29}{2}.$$

Whence, $$x = 4 \text{ or } -25.$$

15. $$\frac{3\sqrt{x}-10}{5(x-5)} = \frac{1}{20}.$$

Clearing of fractions, $$60\sqrt{x}-200 = 5x-25.$$

Transposing and reducing, $$12\sqrt{x} = x+35.$$

Squaring, $$144x = x^2+70x+1225.$$

Transposing and uniting, $$x^2-74x = -1225.$$

Adding $(37)^2$, $$x^2-74x+1369 = 144.$$

Extracting square root, $$x-37 = \pm 12.$$

Whence, $$x = 37 \pm 12 = 49 \text{ or } 25.$$

11. $$\sqrt{6x+1}+\sqrt{x+4+\sqrt{6x+1}} = 2.$$

$$\sqrt{x+4+\sqrt{6x+1}} = 2-\sqrt{6x+1}.$$

Squaring, $$x+4+\sqrt{6x+1} = 4-4\sqrt{6x+1}+6x+1.$$

Transposing and uniting, $$5\sqrt{6x+1} = 5x+1.$$

Squaring, $$150x+25 = 25x^2+10x+1.$$

Transposing and uniting, $$25x^2-140x = 24.$$

Adding 14^2, $$25x^2-140x+196 = 220.$$

Extracting square root, $$5x-14 = \pm 2\sqrt{55}.$$

Whence, $$x = \tfrac{2}{5}(7 \pm \sqrt{55}).$$

12. $$\sqrt{x+5} = \frac{12}{\sqrt{x+12}}.$$

Squaring, $$x+5 = \frac{144}{x+12}.$$

Clearing of fractions, $$x^2+17x+60 = 144.$$

Transposing and uniting, $$x^2+17x = 84.$$

Adding $(\frac{17}{2})^2$, $$x^2+17x+\tfrac{289}{4} = \tfrac{625}{4}.$$

Extracting square root, $$x+\tfrac{17}{2} = \pm \tfrac{25}{2}.$$

Whence, $$x = -\tfrac{17}{2} \pm \tfrac{25}{2} = 4 \text{ or } -21.$$

13. $$\frac{5(3x-1)}{5\sqrt{x}+1}+\frac{2}{\sqrt{x}} = 3\sqrt{x}.$$

Clearing of fractions,

$$15\sqrt{x^3}-5\sqrt{x}+10\sqrt{x}+2 = 15\sqrt{x^3}+3x.$$

Transposing and uniting, $$5\sqrt{x} = 3x-2.$$

Squaring, $$25x = 9x^2-12x+4.$$

Transposing and uniting, $$9x^2-37x = -4.$$

Adding $(\frac{37}{6})^2$, $$9x^2-37x+\tfrac{1369}{36} = \tfrac{1225}{36}.$$

Extracting square root, $$3x-\tfrac{37}{6} = \pm\tfrac{35}{6}.$$

Whence, $$x = 4 \text{ or } \tfrac{1}{9}.$$

14. $$\sqrt{2x+1}+2\sqrt{x} = \frac{21}{\sqrt{2x+1}}.$$

Clearing of fractions,

$$2x+1+2\sqrt{2x^2+x} = 21.$$

Transposing and reducing, $$\sqrt{2x^2+x} = 10-x.$$

Squaring, $$2x^2+x = 100-20x+x^2.$$

Transposing and uniting, $$x^2+21x = 100.$$

Adding $(\frac{21}{2})^2$, $$x^2+21x+\tfrac{441}{4} = \tfrac{841}{4}.$$

Extracting square root, $$x+\tfrac{21}{2} = \pm\tfrac{29}{2}.$$

Whence, $$x = 4 \text{ or } -25.$$

15. $$\frac{3\sqrt{x}-10}{5(x-5)} = \frac{1}{20}.$$

Clearing of fractions, $$60\sqrt{x}-200 = 5x-25.$$

Transposing and reducing, $$12\sqrt{x} = x+35.$$

Squaring, $$144x = x^2+70x+1225.$$

Transposing and uniting, $$x^2-74x = -1225.$$

Adding $(37)^2$, $$x^2-74x+1369 = 144.$$

Extracting square root, $$x-37 = \pm 12.$$

Whence, $$x = 37 \pm 12 = 49 \text{ or } 25.$$

25.
$$\frac{\sqrt{x^2-16}}{\sqrt{x-3}}+\sqrt{x+3}=\frac{7}{\sqrt{x-3}}.$$
$$\frac{\sqrt{x^2-16}}{\sqrt{x-3}}=\frac{7}{\sqrt{x-3}}-\sqrt{x+3}.$$

Subtracting terms of second member,
$$\frac{\sqrt{x^2-16}}{\sqrt{x-3}}=\frac{7-\sqrt{x^2-9}}{\sqrt{x-3}}.$$

Clearing of fractions, $\sqrt{x^2-16}=7-\sqrt{x^2-9}.$

Squaring, $x^2-16=49-14\sqrt{x^2-9}+x^2-9.$

Transposing and uniting, $14\sqrt{x^2-9}=56,$

or, $\sqrt{x^2-9}=4.$

Squaring, $x^2-9=16.$

Whence, $x=\pm 5.$

26.
$$2\sqrt{x}+\frac{2}{\sqrt{x}}=5.$$

Clearing of fractions, $2x+2=5\sqrt{x}.$

Squaring, $4x^2+8x+4=25x.$

Transposing and uniting, $4x^2-17x=-4.$

Whence, $x=4$ or $\frac{1}{4}.$

27.
$$\sqrt{x+8}-\sqrt{x+3}=\sqrt{x}.$$
$$\sqrt{x+8}=\sqrt{x}+\sqrt{x+3}.$$

Squaring, $x+8=x+2\sqrt{x^2+3x}+x+3.$

Transposing and uniting, $5-x=2\sqrt{x^2+3x}.$

Squaring, $25-10x+x^2=4x^2+12x.$

Transposing and uniting, $3x^2+22x=25,$

Whence, $x=1$ or $-\frac{25}{3}.$

28. $$\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}.$$

Squaring, $x+3+2\sqrt{x^2+11x+24}+x+8=25x.$

Transposing and uniting,
$$2\sqrt{x^2+11x+24}=23x-11.$$

Squaring, $$4x^2+44x+96=529x^2-506x+121.$$

Transposing and uniting, $525x^2-550x=-25,$

or, $$21x^2-22x=-1.$$

Whence, $$x=1 \text{ or } \tfrac{1}{21}.$$

29. $$\sqrt{2x+9}+\sqrt{3x-15}=\sqrt{7x+8}$$

Squaring,
$$2x+9+2\sqrt{6x^2-3x-135}+3x-15=7x+8.$$

Transposing and uniting,
$$2\sqrt{6x^2-3x-135}=2x+14,$$

or, $$\sqrt{6x^2-3x-135}=x+7.$$

Squaring, $$6x^2-3x-135=x^2+14x+49.$$

Transposing and uniting, $5x^2-17x=184.$

Whence, $$x=8 \text{ or } -\tfrac{23}{5}.$$

30. $$\sqrt{x+4}-\sqrt{x}=\sqrt{x+1\tfrac{1}{3}}.$$

Squaring, $$x+4-2\sqrt{x^2+4x}+x=x+1\tfrac{1}{3}.$$

Transposing and uniting, $x+2\tfrac{2}{3}=2\sqrt{x^2+4x}.$

Clearing of fractions, $2x+5=4\sqrt{x^2+4x}.$

Squaring, $$4x^2+20x+25=16x^2+64x.$$

Transposing and uniting, $12x^2+44x=25.$

Whence, $$x=\tfrac{1}{2} \text{ or } -4\tfrac{1}{6}.$$

31. $$\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1.$$

$$\sqrt{x-\sqrt{1-x}}=1-\sqrt{x}.$$

Squaring, $$x-\sqrt{1-x}=1-2\sqrt{x}+x.$$

Transposing and uniting, $-\sqrt{1-x} = 1-2\sqrt{x}.$

Squaring, $1-x = 1-4\sqrt{x}+4x.$

Transposing and uniting, $5x = 4\sqrt{x}.$

Squaring, $25x^2-16x = 0.$

Whence, $x = \frac{16}{25}$ or $0.$

32.
$$\frac{\sqrt{1+x}}{1+\sqrt{1+x}} = \frac{\sqrt{1-x}}{1-\sqrt{1-x}}.$$

By (1), $$\frac{1+2\sqrt{1+x}}{\sqrt{1+x}} = \frac{1}{\sqrt{1-x}}.$$

Clearing of fractions, $\sqrt{1+x} = \sqrt{1-x}+2\sqrt{1-x^2},$

or, $\sqrt{1+x}-\sqrt{1-x} = 2\sqrt{1-x^2}.$

Squaring, $1+x-2\sqrt{1-x^2}+1-x = 4-4x^2.$

Transposing and uniting, $2\sqrt{1-x^2} = 2-4x^2,$

or, $\sqrt{1-x^2} = 1-2x^2.$

Squaring, $1-x^2 = 1-4x^2+4x^4.$

Whence, $x = \pm\sqrt{\frac{3}{4}}$ or $\pm\frac{1}{2}\sqrt{3}.$

33.
$$2x+2\sqrt{x+2} = 3\tfrac{1}{2}.$$

Clearing of fractions and transposing,

$$4x-7 = -4\sqrt{x+2}.$$

Squaring, $16x^2-56x+49 = 16x+32.$

Transposing and uniting, $16x^2-72x = -17.$

Whence, $x = 4\frac{1}{4}$ or $\frac{1}{4}.$

34.
$$\frac{3x-1}{\sqrt{3x}+1} = 1+\frac{\sqrt{3x}-1}{2}.$$

Clearing of fractions, $6x-2 = 3x+2\sqrt{3x}+1.$

Transposing and uniting, $3x-3 = 2\sqrt{3x}.$

Squaring, $9x^2-18x+9 = 12x.$

Transposing and uniting, $9x^2-30x = -9.$

Whence, $x = 3$ or $\frac{1}{3}.$

35. $$\frac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=\frac{x}{a-x}.$$

By (1), $$\frac{2a}{a-\sqrt{2ax-x^2}}=\frac{a}{x}.$$

Dividing numerators by a and clearing of fractions,

$$2x=a-\sqrt{2ax-x^2}.$$

Transposing, $$\sqrt{2ax-x^2}=a-2x.$$

Squaring, $$2ax-x^2=a^2-4ax+4x^2.$$

Transposing and uniting, $$5x^2-6ax=-a^2.$$

Whence, $$x=a \text{ or } \frac{a}{5}.$$

36. $$\sqrt{5a+x}+\sqrt{5a-x}=\frac{12a}{\sqrt{5a+x}}.$$

Clearing of fractions,

$$5a+x+\sqrt{25a^2-x^2}=12a.$$

Transposing and uniting, $$\sqrt{25a^2-x^2}=7a-x.$$

Squaring, $$25a^2-x^2=49a^2-14ax+x^2$$

Transposing and uniting, $$2x^2-14ax=-24a^2.$$

Whence, $$x=4a \text{ or } 3a.$$

37. $$\frac{\sqrt{a+x}}{\sqrt{a}-\sqrt{a+x}}=\frac{\sqrt{a-x}}{\sqrt{x}-\sqrt{a-x}}.$$

By (1), $$\frac{\sqrt{a}}{\sqrt{a+x}}=\frac{\sqrt{x}}{\sqrt{a-x}}.$$

Squaring, $$\frac{a}{a+x}=\frac{x}{a-x}.$$

Clearing of fractions, $$a^2-ax=ax+x^2.$$

Transposing and uniting, $$x^2+2ax=a^2.$$

Whence, $$x=-a\pm a\sqrt{2} \text{ or } a(-1\pm\sqrt{2}).$$

38. $$\sqrt{2x+2}=\sqrt{7x}-3.$$

Squaring, $$2x+2=7x-6\sqrt{7x}+9.$$

Transposing and uniting, $$6\sqrt{7x}=5x+7.$$

Squaring, $$252x=25x^2+70x+49.$$

Transposing and uniting, $$25x^2-182x=-49.$$

Whence, $$x=7 \text{ or } \tfrac{7}{25}.$$

39.

$$2\sqrt{x}+\sqrt{4x+\sqrt{7x+2}}=1.$$

$$\sqrt{4x+\sqrt{7x+2}}=1-2\sqrt{x}.$$

Squaring, $4x+\sqrt{7x+2}=1-4\sqrt{x}+4x.$

Transposing and uniting, $\sqrt{7x+2}=1-4\sqrt{x}.$

Squaring. $7x+2=1-8\sqrt{x}+16x.$

Transposing and uniting, $8\sqrt{x}=9x-1.$

Squaring, $64x=81x^2-18x+1.$

Transposing and uniting, $81x^2-82x=-1.$

Whence, $x=1 \text{ or } \frac{1}{81}.$

40.

$$\frac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x+2a}+\sqrt{x-2a}}=\frac{x}{2a}.$$

By (7), $\dfrac{2\sqrt{x+2a}}{-2\sqrt{x-2a}}=\dfrac{x+2a}{x-2a},$

or, $\dfrac{\sqrt{x+2a}}{\sqrt{x-2a}}=\dfrac{x+2a}{x-2a}.$

Squaring, $\dfrac{x+2a}{x-2a}=\dfrac{x^2+4ax+4a^2}{x^2-4ax+4a^2}.$

By (7), $\dfrac{2x}{4a}=\dfrac{2x^2+8a^2}{8ax},$

or, $\dfrac{x}{2a}=\dfrac{x^2+4a^2}{4ax}.$

Clearing of fractions, $2x^2=x^2+4a^2.$

Extracting square root, $x=\pm 2a.$

Art. 321.

1. $3x^2-15x=-18.$

Dividing by 3, $x^2-5x=-6.$

Multiplying by 4, $4x^2-20x=-24.$

Adding 5^2, $4x^2-20x+25=1.$

Extracting square root, $2x-5=\pm 1.$

Whence, $x=3 \text{ or } 2.$

2. $2x^2 - 6x = 20.$

Dividing by 2, $x^2 - 3x = 10.$

Multiplying by 4, $4x^2 - 12x = 40.$

Adding 3^2, $4x^2 - 12x + 9 = 49.$

Extracting square root, $2x - 3 = \pm 7.$

Whence, $x = 5$ or $-2.$

3. $3x^2 - 22x = -7.$

Multiplying by 3, $9x^2 - 66x = -21.$

Multiplying by 4, $36x^2 - 264x = -84.$

Adding 22^2, $36x^2 - 264x + 484 = 400.$

Extracting square root, $6x - 22 = \pm 20.$

Whence, $x = 7$ or $\frac{1}{3}.$

4. $10x^2 - x = 21.$

Multiplying by 10, $100x^2 - 10x = 210.$

Multiplying by 4, $400x^2 - 40x = 840.$

Adding 1^2, $400x^2 - 40x + 1 = 841.$

Extracting square root, $20x - 1 = \pm 29.$

Whence, $x = \frac{3}{2}$ or $-1\frac{2}{5}.$

5. $12x^2 + 13x = 35.$

Multiplying by 12, $144x^2 + 156x = 420.$

Multiplying by 4, $576x^2 + 624x = 1680.$

Adding 169, $576x^2 + 624x + 169 = 1849.$

Extracting square root, $24x + 13 = \pm 43.$

Whence, $x = 1\frac{1}{4}$ or $-2\frac{1}{3}.$

6. $2x^2 - x = 6.$

Multiplying by 2, $4x^2 - 2x = 12.$

Multiplying by 4, $16x^2 - 8x = 48.$

Adding 1^2, $16x^2 - 8x + 1 = 49.$

Extracting square root, $4x - 1 = \pm 7.$

Whence, $x = 2$ or $-1\frac{1}{2}.$

7. $15x^2+34x=-15.$

Multiplying by 15, $225x^2+510x=-225.$

Multiplying by 4, $900x^2+2040x=-900.$

Adding 34^2, $900x^2+2040x+1156=256.$

Extracting square root, $30x+34=\pm 16.$

Whence, $x=-\frac{3}{5}$ or $-1\frac{2}{3}.$

8. $14x^2+x=175$

Multiplying by 14, $196x^2+14x=2450.$

Multiplying by 4, $784x^2+56x=9800.$

Adding 1^2, $784x^2+56x+1=9801.$

Extracting square root, $28x+1=\pm 99.$

Whence, $x=3\frac{1}{2}$ or $-3\frac{4}{7}.$

9. $3x^2+2x=85.$

Multiplying by 3, $9x^2+6x=255.$

Multiplying by 4, $36x^2+24x=1020.$

Adding 2^2, $36x^2+24x+4=1024.$

Extracting square root, $6x+2=\pm 32.$

Whence, $x=5$ or $-5\frac{2}{3}.$

10. $3x^2+4x=340.$

Multiplying by 3, $9x^2+12x=1020.$

Multiplying by 4, $36x^2+48x=4080.$

Adding 4^2, $36x^2+48x+16=4096.$

Extracting square root, $6x+4=\pm 64.$

Whence, $x=10$ or $-11\frac{1}{3}.$

11. $3x^2-14x=-15.$

Multiplying by 3, $9x^2-42x=-45.$

Multiplying by 4, $36x^2-168x=-180.$

Adding 14^2, $36x^2-168x+196=16.$

Extracting square root, $6x-14=\pm 4.$

Whence, $x=3$ or $1\frac{2}{3}.$

12. $$5x^2+6x = 63.$$

Multiplying by 5, $$25x^2+30x = 315.$$

Multiplying by 4, $$100x^2+120x = 1260.$$

Adding 6^2, $$100x^2+120x+36 = 1296.$$

Extracting square root, $$10x+6 = \pm 36.$$

Whence, $$x = 3 \text{ or } -4\tfrac{1}{5}.$$

13. $$4x^2-6x = 108.$$

Multiplying by 4, $$16x^2-24x = 432.$$

Multiplying by 4, $$64x^2-96x = 1728.$$

Adding 6^2, $$64x^2-96x+36 = 1764.$$

Extracting square root, $$8x-6 = \pm 42.$$

Whence, $$x = 6 \text{ or } -4\tfrac{1}{2}.$$

14. $$3x^2-2x = 65.$$

Multiplying by 3, $$9x^2-6x = 195.$$

Multiplying by 4, $$36x^2-24x = 780.$$

Adding 2^2, $$36x^2-24x+4 = 784.$$

Extracting square root, $$6x-2 = \pm 28.$$

Whence, $$x = 5 \text{ or } -4\tfrac{1}{3}.$$

15. $$7x^2-20x = 32.$$

Multiplying by 7, $$49x^2-140x = 224.$$

Multiplying by 4, $$196x^2-560x = 896.$$

Adding 20^2, $$196x^2-560x+400 = 1296.$$

Extracting square root, $$14x-20 = \pm 36.$$

Whence, $$x = 4 \text{ or } -1\tfrac{1}{7}.$$

16. $$5x^2+4x = 273.$$

Multiplying by 5, $$25x^2+20x = 1365.$$

Multiplying by 4, $$100x^2+80x = 5460.$$

Adding 4^2, $$100x^2+80x+16 = 5476.$$

Extracting square root, $$10x+4 = \pm 74.$$

Whence, $$x = 7 \text{ or } -7\tfrac{4}{5}.$$

17. $$3x^2+4x = 95.$$

Multiplying by 3, $$9x^2+12x = 285.$$

Multiplying by 4, $$36x^2+48x = 1140.$$

Adding 4^2, $$36x^2+48x+16 = 1156.$$

Extracting square root, $$6x+4 = \pm 34.$$

Whence, $$x = 5 \text{ or } -6\tfrac{1}{3}.$$

18. $$11x^2-100x = -201.$$

Multiplying by 11, $$121x^2-1100x = -2211.$$

Adding 50^2, $$121x^2-1100x+2500 = 289.$$

Extracting square root, $$11x-50 = \pm 17.$$

Whence, $$x = 6\tfrac{1}{11} \text{ or } 3.$$

19. $$\frac{x^2}{3}+\frac{4x}{5} = 34\tfrac{1}{5}.$$

Clearing of fractions, $$5x^2+12x = 513.$$

Multiplying by 5, $$25x^2+60x = 2565.$$

Multiplying by 4, $$100x^2+240x = 10260.$$

Adding 12^2, $$100x^2+240x+144 = 10404.$$

Extracting square root, $$10x+12 = \pm 102.$$

Whence, $$x = 9 \text{ or } -11\tfrac{2}{5}.$$

20. $$\frac{x^2}{5}+20x+80 = 3x^2.$$

Clearing of fractions, $$x^2+100x+400 = 15x^2.$$

Transposing and uniting, $$14x^2-100x = 400,$$

or, $$7x^2-50x = 200.$$

Multiplying by 7, $$49x^2-350x = 1400.$$

Multiplying by 4, $$196x^2-1400x = 5600.$$

Adding 50^2, $$196x^2-1400x+2500 = 8100.$$

Extracting square root, $$14x-50 = \pm 90.$$

Whence, $$x = 10 \text{ or } -2\tfrac{6}{7}.$$

21. $$\frac{x}{5}+\frac{5}{x} = \frac{26}{5}.$$

Clearing of fractions, $$x^2+25 = 26x.$$

Transposing, $x^2-26x=-25.$

Multiplying by 4, $4x^2-104x=-100.$

Adding 26^2, $4x^2-104x+676=576.$

Extracting square root, $2x-26=\pm 24.$

Whence, $x=25$ or 1.

22. $11\frac{3}{4}x-3\frac{1}{2}x^2=-41\frac{1}{4}.$

Clearing of fractions, $47x-14x^2=-165,$

or, $14x^2-47x=165.$

Multiplying by 14, $196x^2-658x=2310.$

Multiplying by 4, $784x^2-2592x=9240.$

Adding 47^2, $784x^2-2592x+2209=11449.$

Extracting square root, $28x-47=\pm 107.$

Whence, $x=5\frac{1}{2}$ or $-2\frac{1}{7}.$

23. $\frac{x^2}{81}-\frac{2x}{9}=-5.$

Clearing of fractions, $x^2-18x=-405.$

Multiplying by 4, $4x^2-72x=-1620.$

Adding 18^2, $4x^2-72x+324=-1296.$

Extracting square root, $2x-18=\pm 36\sqrt{-1}.$

Whence, $x=9(1\pm 2\sqrt{-1}).$

24. $\frac{9x^2}{16}+3\frac{3}{4}x=-4.$

Clearing of fractions, $9x^2+60x=-64.$

Multiplying by 4, $36x^2+240x=-256.$

Adding 20^2, $36x^2+240x+400=144.$

Extracting square root, $6x+20=\pm 12.$

Whence, $x=-1\frac{1}{3}$ or $-5\frac{1}{3}.$

25. $\frac{4x^2}{9}-4x=7.$

Clearing of fractions, $4x^2-36x=63.$

Adding 81, $4x^2-36x+81=144.$

Extracting square root, $2x-9=\pm 12.$

Whence, $x=10\frac{1}{2}$ or $-1\frac{1}{2}.$

26. $\frac{4x^2}{49}+\frac{8x}{21}=6\frac{2}{3}.$

Clearing of fractions, $12x^2+56x=980,$

or, $3x^2+14x=245.$

Multiplying by 3, $9x^2+42x=735.$

Multiplying by 4, $36x^2+168x=2940.$

Adding 14^2, $36x^2+168x+196=3136.$

Extracting square root, $6x+14=\pm 56.$

Whence, $x=7 \text{ or } -11\frac{2}{3}.$

27. $\frac{x^2}{361}-\frac{12x}{19}=-32.$

Clearing of fractions, $x^2-228x=-11552.$

Multiplying by 4, $4x^2-912x=-46208.$

Adding $(228)^2$, $4x^2-912x+51984=5776.$

Extracting square root, $2x-228=\pm 76.$

Whence, $x=152 \text{ or } 76.$

Art. 323.

1. $ax^2-bx=c.$

Multiplying by $4a$, $4a^2x^2-4abx=4ac.$

Adding b^2, $4a^2x^2-4abx+b^2=4ac+b^2.$

Extracting square root, $2ax-b=\pm\sqrt{4ac+b^2}.$

Transposing, $2ax=b\pm\sqrt{4ac+b^2}.$

Whence, $x=\frac{b\pm\sqrt{4ac+b^2}}{2a}.$

2. $cx^2+2ax=b.$

Multiplying by $4c$, $4c^2x^2+8acx=4bc.$

Adding $4a^2$, $4c^2x^2+8acx+4a^2=4bc+4a^2.$

Extracting square root, $2cx+2a=\pm 2\sqrt{bc+a^2},$

or, $cx+a=\pm\sqrt{bc+a^2}.$

Transposing. $cx=-a\pm\sqrt{bc+a^2}.$

Whence, $x=\frac{-a\pm\sqrt{bc+a^2}}{c}.$

3. $$(x+a)^2 = 5ax-(x-a)^2.$$

Simplifying, $$2x^2-5ax = -2a^2.$$

Multiplying by 8, $$16x^2-40ax = -16a^2.$$

Adding $25a^2$, $$16x^2-40ax+25a^2 = 9a^2.$$

Extracting square root, $$4x-5a = \pm 3a.$$

Transposing, $$4x = 5a \pm 3a.$$

Whence, $$x = 2a \text{ or } \frac{a}{2}.$$

4. $$10x^2-9ax = -9a^2.$$

Multiplying by 40, $$400x^2-360ax = -360a^2.$$

Adding $81a^2$, $$400x^2-360ax+81a^2 = -279a^2.$$

Extracting square root, $$20x-9a = \pm a\sqrt{-279}.$$

Transposing, $$20x = 9a \pm a\sqrt{-279}.$$

Whence, $$x = \frac{9a \pm a\sqrt{-279}}{20}.$$

5. $$(a+b)x^2+(a-b)x = \frac{ab}{a+b}.$$

Clearing of fractions, $$(a+b)^2x^2+(a^2-b^2)x = ab.$$

Multiplying by 4, $$4(a+b)^2x^2+4(a^2-b^2)x = 4ab.$$

Adding $(a-b)^2$,
$$4(a+b)^2x^2+4(a^2-b^2)x+(a-b)^2 = a^2+2ab+b^2.$$

Extracting square root, $$2(a+b)x+(a-b) = \pm(a+b).$$

Whence, $$x = \frac{b}{a+b} \text{ or } -\frac{a}{a+b}.$$

6. $$\frac{2x(a-x)}{3a-2x} = \frac{a}{4}.$$

Clearing of fractions, $$8ax-8x^2 = 3a^2-2ax.$$

Transposing and uniting, $$8x^2-10ax = -3a^2.$$

Multiplying by 8, $$64x^2-80ax = -24a^2.$$

Adding $25a^2$, $$64x^2-80ax+25a^2 = a^2.$$

Extracting square root, $$8x-5a = \pm a.$$

Whence, $$x = \frac{3a}{4} \text{ or } \frac{a}{2}.$$

7. $$\frac{1}{x+a}+\frac{1}{x+2a}+\frac{1}{x+3a} = \frac{3}{x}.$$

Clearing of fractions,

$$3x^3+12ax^2+11a^2x = 3x^3+18ax^2+33a^2x+18a^3,$$

or, $$6ax^2+22a^2x = -18a^3.$$

$$3x^2+11ax = -9a^2.$$

Multiplying by 12, $$36x^2+132ax = -108a^2.$$

Adding $121a^2$, $$36x^2+132ax+121a^2 = 13a^2.$$

Extracting square root, $$6x+11a = \pm a\sqrt{13}.$$

Whence, $$x = \frac{-11a \pm a\sqrt{13}}{6}.$$

8. $$(a+b)x^2-cx = \frac{ac}{a+b}.$$

Multiplying by $4(a+b)$,

$$4(a+b)^2x^2-4(a+b)cx = 4ac.$$

Adding c^2, $$4(a+b)^2x^2-4(a+b)cx+c^2 = 4ac+c^2.$$

Extracting square root, $$2(a+b)x-c = \pm\sqrt{c^2+4ac}.$$

Whence, $$x = \frac{c \pm \sqrt{c^2+4ac}}{2(a+b)}.$$

9. $$(a-c)x^2-bx = c.$$

Multiplying by $4(a-c)$,

$$4(a-c)^2x^2-4(a-c)bx = 4c(a-c).$$

Adding b^2, $$4(a-c)^2x^2-4(a-c)bx+b^2 = 4c(a-c)+b^2.$$

Extracting square root, $$2(a-c)x-b = \pm\sqrt{b^2+4c(a-c)}.$$

Whence, $$x = \frac{b \pm \sqrt{b^2+4c(a-c)}}{2(a-c)}.$$

10. $$\frac{2c^2}{d^2}+\frac{ac}{d}-(a-b)(2c+ad)\frac{x}{d}=(a+b)\frac{cx}{d}-(a^2-b^2)x^2.$$

Simplifying,

$$\frac{c(2c+ad)-d(a-b)(2c+ad)x}{d^2}=\frac{(a+b)cdx-(a^2-b^2)d^2x^2}{d^2}.$$

Dividing by $(2c+ad)$, $$\frac{c-d(a-b)x}{d^2}=\frac{(a+b)cdx-(a^2-b^2)d^2x^2}{d^2(2c+ad)}.$$

Dividing by $(a+b)$, $$\frac{c-d(a-b)x}{d^2(a+b)}=\frac{dx[c-d(a-b)x]}{d^2(2c+ad)}.$$

Dividing by $\dfrac{d^2}{c-d(a-b)x}$, $$\frac{1}{a+b}=\frac{dx}{2c+ad}.$$

Clearing of fractions, $$(a+b)dx=2c+ad.$$

Whence, $$x=\frac{2c+ad}{d(a+b)}.$$

See Appendix.

11. $$abx^2+\frac{3a^2x}{c}+\frac{b^2x}{c}=\frac{(3a+2b)(2a-b)}{c^2}.$$

Clearing of fractions,

$$abc^2x^2+3a^2cx+b^2cx=(3a+2b)(2a-b).$$

$$(abc^2)x^2+(3a^2c+b^2c)x=6a^2+ab-2b^2.$$

Multiplying by $4ab$,

$$4a^2b^2c^2x^2+(12a^3bc+4ab^3c)x=24a^3b+4a^2b^2-8ab^3.$$

Adding $(3a^2+b^2)^2$,

$$4a^2b^2c^2x^2+(12a^3bc+4ab^3c)x+9a^4+6a^2b^2+b^4=9a^4+10a^2b^2+24a^3b-8ab^3+b^4.$$

Extracting square root,

$$2abcx+(3a^2+b^2)=\pm(3a^2+4ab-b^2).$$

Whence, $$x=\frac{2a-b}{ac}\text{ or }-\frac{3a+2b}{bc}.$$

12. $$a^2x^2-2a^3x+a^4-1=0.$$

$$a^2x^2-2a^3x=1-a^4.$$

Dividing by a^2, $$x^2-2ax=\frac{1-a^4}{a^2}.$$

Adding a^2, $$x^2-2ax+a^2=\frac{1}{a^2}.$$

Extracting square root, $$x-a = \pm\frac{1}{a}.$$

Whence, $$x = \frac{a^2 \pm 1}{a}.$$

13. $$(3a^2+b^2)(x^2-x+1) = (3b^2+a^2)(x^2+x+1).$$

$$3a^2x^2-3a^2x+3a^2+b^2x^2-b^2x+b^2 = 3b^2x^2+3b^2x+3b^2+a^2x^2+a^2x+a^2$$

$$(a^2-b^2)x^2-2(a^2+b^2)x = -(a^2-b^2).$$

Multiplying by (a^2-b^2),

$$(a^2-b^2)^2x^2-2(a^4-b^4)x = -(a^4-2a^2b^2+b^4).$$

Adding $(a^2+b^2)^2$,

$$(a^2-b^2)^2x^2-2(a^4-b^4)x+(a^2+b^2)^2 = 4a^2b^2.$$

Extracting square root,

$$(a^2-b^2)x-(a^2+b^2) = \pm 2ab.$$

Whence, $$x = \frac{a+b}{a-b} \text{ or } \frac{a-b}{a+b}.$$

See Appendix.

Art. 328.

1. Let $2x =$ first number, and $3x =$ second number.

Then, $9x^2-4x^2 = 125.$

Whence, $5x^2 = 125,$

and $x = \pm 5.$

Hence, $3x = \pm 15,$ and $2x = \pm 10.$

2. Let $6x =$ Mr. Brown's age, and $5x =$ Mrs. Brown's age.

Then $36x^2+25x^2 = 1525.$

Whence, $61x^2 = 1525,$

$x^2 = 25,$

and $x = 5.$

Hence, $6x = 30$ yrs., and $5x = 25$ yrs.

3. Let $x =$ width of hall, and $2\frac{1}{2}x =$ length of hall.

Then will $2\frac{1}{2}x^2 = 4000$ sq. ft.

Whence, $x^2 = 1600,$

and $x = 40$ ft.

Hence, $2\frac{1}{2}x = 100$ ft.

4. Let $15x$ = No. yards of cloth, and $4x$ = No. dollars per yard.

Then will $\quad 4x \times 15x = 163.35.$

Whence, $\quad x^2 = 2.722+,$

and, $\quad x = 1.65-.$

Hence, $\quad 4x = \$6\frac{3}{5}$, and $15x = 24\frac{3}{4}$ yds.

5. Let x = greater number, and $\frac{126}{x}$ = less number.

Then will $$\frac{x}{\frac{126}{x}} = 3\tfrac{1}{2}.$$

$$\frac{x^2}{126} = \frac{7}{2}, \quad \text{or} \quad x^2 = 441.$$

Whence, $\quad x = 21.$

Hence, $\quad \frac{126}{x} = 6.$

6. Let $2x$ = first number, $3x$ = second number,

and $5x$ = third number.

Then will $\quad 4x^2 + 9x^2 + 25x^2 = 1862.$

Whence, $\quad 38x^2 = 1862,$

$x^2 = 49,$

and $\quad x = 7.$

Hence, $\quad 2x = 14, \quad 3x = 21, \quad 5x = 35.$

7. Let $8x$ = No. weeks, and $7x$ = No. dollars earned per week.

Then will $\quad 56x^2 = 1400.$

Whence, $\quad 4x^2 = 100,$

$2x = 10,$

and $\quad x = 5.$

Hence, $\quad 8x = 40$ weeks, and $7x = \$35.$

8. Let x = the number.

Then will $\quad 3x^2 - 5x = 7x.$

Whence, $\quad 3x^2 = 12,$

$x^2 = 4$, and $x = \pm 2.$

9. Let x = the number.

Then will $$\frac{1}{8}\left(\frac{x}{7}\times\frac{x}{8}\right) = 298\tfrac{2}{3}.$$

Whence, $$\frac{x^2}{56} = 896.$$

$$x^2 = 50176, \quad x = \pm 224.$$

10. Let x = number of apples, and $80-x$ = number of pears.

$$\frac{10}{80-x} = \text{cost of one apple, and } \frac{45}{x} = \text{cost of one pear.}$$

Then $$\frac{5x}{80-x} = \frac{(80-x)45}{x}.$$

Clearing of fractions and reducing,

$$x^2-180x = -7200.$$

Whence, $$x = 60 \text{ or } 120,$$

and $$80-x = 20 \text{ or } -40.$$

11. Let x = number of men in front, $x+5$ = number of men in depth, and $x(x+5)$ = whole number of men.

Then will $$5(x+845) = x(x+5).$$

$$5x+4225 = x^2+5x.$$

$$x^2 = 4225.$$

Whence, $$x = 65,$$

$$x+5 = 70, \quad \text{and} \quad x(x+5) = 4550.$$

12. Let x = amount of principal, and $.04x$ = interest for 8 mo. at 6%.

Then will $$.04x\,(x) = \$900.$$

Whence, $$.04x^2 = \$900,$$

$$.2x = \$30, \quad \text{and} \quad x = \$150.$$

13. Let $$4x = \text{number of rds. in length of field,}$$

$$3x = \text{number of rds. in width of field,}$$

and $$4x\times 3x = \text{number of sq. rds. in field.}$$

Then, since the square of the hypothenuse of a right-angled triangle is equal to the sum of the squares of the other two sides,

$$16x^2+9x^2 = 10000.$$

Whence, $$25x^2 = 10000,$$

$$x^2 = 400,$$

and $$x = 20.$$

Hence, $$4x = 80, \qquad 3x = 60,$$

and $$4x \times 3x = 4800 \text{ sq. rds.}$$

$$4800 \text{ sq. rds.} = 30 \text{ acres.}$$

14. Let x = number of rds. in a side of each of the three lots,

and $$3x^2 = \text{their total area.}$$

Then will $$3x^2 + 193 = 25 \times 25.$$

Whence, $$4x^2 = 432,$$

$$x^2 = 144, \qquad x = 12 \text{ rds.}$$

15. Let x = less number, and $17-x$ = greater number.

Then will $$\frac{x}{17-x} : \frac{17-x}{x} :: 64 : 81.$$

Whence, $$\frac{81x}{17-x} = \frac{64(17-x)}{x},$$

$$81x^2 = 64(17-x)^2,$$

$$9x = 8(17-x),$$

$$17x = 136, \quad \text{and} \quad x = 8.$$

Hence, $$17-x = 9.$$

Art. 329.

1. Let x = first part, and $56-x$ = second part.

Then will $$56x - x^2 = 640.$$

$$x^2 - 56x = -640.$$

Hence, $$x = 28 \pm \sqrt{28^2 - 640}.$$

$$x = 40, \quad \text{and} \quad 56-x = 16.$$

2. Let x = first number, and $x+6$ = second number.

Then will $$x^2 + (x+6)^2 = 50.$$

$$x^2 + x^2 + 12x + 36 = 50.$$

$$x^2 + 6x = 7.$$

Hence, $$x = -3 \pm \sqrt{9+7},$$

$$x = 1, \quad \text{and} \quad x+6 = 7.$$

3. Let $x =$ number of trees in depth,

and $x+50 =$ number of trees in front.

Then will $x(x+50) = 8400.$

$$x^2+50x = 8400.$$

Hence, $x = -25 \pm \sqrt{625+8400},$

$$x = 70.$$

4. Let $x =$ the number of persons;

then $\dfrac{144}{x} =$ number of dollars each received,

and $\dfrac{144}{x+2} =$ number of dollars each received when there were two persons more.

Hence, $\dfrac{144}{x+3} + 1 = \dfrac{144}{x}.$

$$144x + x^2 + 2x = 144x + 288.$$

$$x^2 + 2x = 288.$$

Whence, $x = -1 \pm \sqrt{1+288},$

$$x = 16.$$

5. Let $x =$ number of persons,

and $\dfrac{110}{x} =$ number of bushels each received.

Then will $\dfrac{110}{x} + 1 = x.$

$$110 + x = x^2.$$

$$x^2 - x = 110.$$

$$4x^2 - 4x = 440.$$

$$4x^2 - 4x + 1 = 441.$$

$2x - 1 = 21.$ Whence, $x = 11.$

6. Let $x =$ No. men, and $\dfrac{175}{x} =$ No. bushels each would husk.

Then will $\dfrac{175}{x-2} = \dfrac{175}{x} + 10.$

$$175x = 175x - 350 + 10x^2 - 20x.$$

$$x^2 - 2x = 35.$$

$$x^2 - 2x + 1 = 36.$$

Whence, $x - 1 = 6,$ and $x = 7.$

7. Let x = No. pounds that she bought, and $\frac{\$2.16}{x}$ = cost per pound.

Then will $$\frac{\$2.16}{x} + \$.01 = \frac{\$2.16}{x-3}.$$

$$\$2.16 - \$6.48 + \$.01x^2 - \$.03x = \$2.16x.$$

$$\$.01x^2 - \$.03x = \$6.48.$$

$$x^2 - 3x = 648.$$

$$4x^2 - 12x + 9 = 2601.$$

$$2x - 3 = 51.$$

Whence, $$x = 27.$$

8. Let x = number of dollars per yard, and $\frac{\$45}{x}$ = number of yards.

Then will $$\left(\frac{45}{x} - 5\right)(x + .15) = 49.50.$$

Simplifying, $$5x^2 + 5.25x = 6.75,$$

or, $$x^2 + 1.05x = 1.35.$$

Completing square and extracting square root,

$$x + .525 = 1.275.$$

Whence, $$x = .75, \quad \text{and} \quad \frac{45}{x} = 60.$$

9. Let x = No. rods in width, and $x+8$ = No. rods in length.

Then will $$x(x+8) = 768.$$

Whence, $$x + 4 = 28,$$

$$x = 24, \quad \text{and} \quad x + 8 = 32.$$

10. Let x = number of dollars paid for horse,

and $312.50 - x$ = number of dollars gained.

Then will $$\frac{100(312.50 - x)}{x} = \frac{1}{10}x.$$

Simplifying, $$312500 - 1000x = x^2.$$

$$x^2 + 1000x = 312500.$$

Extracting square root, $$x^2 + 500 = 750.$$

$$x = 250.$$

11. Let x = B's age, and $x+4$ = A's age.

Then will $$x^2+x^2+8x+16 = 976.$$

Whence, $$x^2+4x = 480.$$
$$x+2 = 22,$$
$$x = 20, \quad \text{and} \quad x+4 = 24.$$

12. Let x = No. sheep bought, and $\dfrac{240}{x}$ = No. dollars paid for 1 sheep.

Then will $$6.75x-240 = 5\left(\frac{240}{x}\right).$$

And $$.45x^2-16x = 80.$$

Whence, $$x = 40.$$

13. Let x = cost of goods in dollars.

Then will $$\frac{x-24}{x} = \frac{x}{100}.$$

Whence, $$x^2-100x = -2400.$$

Completing square and extracting square root,
$$x-50 = \pm 10.$$

Whence, $$x = 40 \text{ or } 60.$$

14. Let $12-x$ = tens' figure, and x = units' figure.
$$x(12-x) = 10x+12-x-16.$$

Whence, $$x^2-3x = 4.$$

Extracting square root, $$x-\tfrac{3}{2} = \tfrac{5}{2}.$$

Whence, $$x = 4, \quad \text{and} \quad 12-x = 8.$$

15. Let x = No. yards, and $\dfrac{120}{x}$ = No. dollars paid per yard.

Then will $$\frac{120}{x-4} = \frac{120}{x} + 1.$$

Whence, $$x^2-4x = 480.$$

Extracting square root, $$x-2 = 22$$

Whence, $$x = 24.$$

16. Let x = cost of first kind, and $x+.02$ = cost of second kind.

Then will $$\frac{1.00}{x+.02}+\frac{5}{2}=\frac{1.00}{x}.$$

Whence, $$2x+5x^2+.10x = 2x+.04,$$

or, $$5x^2+.10x = .04.$$

$$x^2+.02x = .008.$$

Extracting square root, $$x+.01 = .09,$$

$$x = .08, \quad \text{and} \quad x+.02 = .10.$$

17. Let x = number of days in which B can do it alone,

and $x+9$ = number of days in which A can do it alone.

Then will $$\frac{1}{x}+\frac{1}{x+9}=\frac{1}{20}.$$

Whence, $$x^2-31x = 180.$$

Extracting square root, $$x-\tfrac{31}{2} = \tfrac{41}{2}.$$

Hence, $$x = 36 \quad \text{and} \quad x+9 = 45.$$

18. Let x = number of cents per dozen.

Then will $$\frac{25}{x}+\frac{5}{12}=\frac{25}{x-5}.$$

Whence, $$5x^2-25x = 1500,$$

or, $$x^2-5x = 300.$$

Extracting square root, $$x-\tfrac{5}{2} = \tfrac{35}{2}.$$

Whence, $$x = 20.$$

19. Let x = one side of the rectangle, and $222-x$ = the other side.

Then will $$x(222-x) = 12096.$$

Whence, $$x^2-222x = -12096.$$

Extracting square root, $$x-111 = 15.$$

Whence, $$x = 126, \quad \text{and} \quad 222-x = 96.$$

20. Let x = number of dollars in the principal,

and $\dfrac{.05x}{4}$ = number of dollars in the interest for three months.

Then will $$x\left(\frac{.05x}{4}\right)-120 = 2\tfrac{1}{2}x.$$

Whence, $.05x^2-10 = 480,$

or, $x^2-200x = 9600.$

Completing square, etc., $x-100 = 140.$

Whence, $x = 240.$

21. Let $x =$ number of hours in which first will fill it;

and $x-6 =$ number of hours in which second will fill it.

Then will $\frac{1}{x}+\frac{1}{x-6} = \frac{1}{4}.$

Whence, $4x-24+4x = x^2-6x,$

or, $x^2-14x = -24.$

Completing square, etc., $x-7 = 5.$

Whence, $x = 12,$

and $x-6 = 6.$

22. Let $x =$ the first number, and $x+11 =$ the second number.

Then will $\frac{4}{5}x(x+11)+345 = (x+11)^2.$

Whence, $4x^2+44x+1725 = 5x^2+110x+605,$

or, $x^2+66x = 1120.$

Completing square, etc., $x+33 = 47.$

Whence, $x = 14,$ and $x+11 = 25.$

23. Let $x =$ number of miles A traveled per hour,

and $x-2 =$ number of miles B traveled per hour.

Then will $\frac{150}{x}+20 = \frac{150}{x-2}.$

$$150x-300+20x^2-40x = 150x.$$

Whence, $20x^2-40x = 300,$

or, $x^2-2x = 15.$

Completing square, etc., $x-1 = 4.$

Whence, $x = 5,$ and $x-2 = 3.$

24. Let $x =$ No. of days in which A alone can do the work,

and $\dfrac{x+3}{\frac{3}{5}} =$ No. of days in which B alone can do the work.

Then will $$\frac{1}{x}+\frac{\frac{3}{5}}{x+3}=\frac{1}{10}.$$

Whence, $$5x^2-65x = 150,$$

or, $$x^2-13x = 30.$$

Completing square, etc., $$x-\tfrac{13}{2} = \tfrac{17}{2}.$$

Whence, $$x = 15, \quad \text{and} \quad \frac{x+3}{\frac{3}{5}} = 30.$$

25. Let $x =$ No. yds. in length, and $154-x =$ No. yds. in width.

Then will $$x(154-x) = 4840.$$

Whence, $$x^2-154x = -4840.$$

Completing square, etc., $$x-77 = 33.$$

Whence, $$x = 110, \quad \text{and} \quad 154-x = 44.$$

26. Let $x =$ rate of the current.

Then will $$\frac{5}{6+x}+2=\frac{5}{6-x}.$$

Whence, $$2x^2+10x = 72,$$

or, $$x^2+5x = 36.$$

Completing square, etc., $$x+\tfrac{5}{2} = \tfrac{13}{2}.$$

Whence, $$x = 4.$$

27. Let $x =$ number of gallons in first cask.

Then will $.10x =$ price per gallon in first cask.

Let $x+10 =$ number of gallons in second cask.

Then will $.10(x+10) =$ price per gallon in second cask.

Then $$.10x(x)+.10(x+10)(x+10) = 610.$$

Whence, $$.20x^2+2x = 600,$$

or $$x^2+10x = 3000.$$

Completing square, etc., $$x+5 = 55.$$

Whence, $$x = 50, \quad \text{and} \quad x+10 = 60.$$

28. Let x = number of miles faster vessel sails per hour,

and $x-2$ = number of miles slower vessel sails per hour.

Then will $$\frac{1152}{x} = \frac{720}{x-2} + 24.$$

Whence, $$x^2-20x = -96.$$

Completing square, etc., $$x-10 = \pm 2.$$

Whence, $$x = 8 \text{ or } 12.$$

29. Let x = number of miles he can row in still water.

Then will $$\frac{1}{x+1} + \frac{1}{x-1} = \frac{20}{48} \text{ or } \frac{5}{12}.$$

Whence, $$12x+12+12x-12 = 5x^2-5,$$

$$5x^2-24x = 5,$$

or, $$x^2-\tfrac{24}{5}x = 1.$$

Completing square, etc., $$x-\tfrac{12}{5} = \tfrac{13}{5}.$$

Whence, $$x = 5.$$

30. Let x = number of rods in width of field,

and $x+4$ = number of rods in length of field.

Then will $$x(x+4) = 9(160).$$

Whence, $$x^2+4x = 1440.$$

Completing square, etc., $$x+2 = 38.$$

Whence, $$x = 36, \quad \text{and} \quad x+4 = 40.$$

31. Let x = number of chickens, and $x+19$ = number of turkeys.

Then will $$\frac{48}{x+19} = \frac{18}{x} + .35.$$

Whence, $$.35x^2+23.35x = -342.$$

Completing square and extracting square root,

$$.35x - \frac{23.35}{2} = \frac{8.15}{2}.$$

Whence, $$x = 45, \quad \text{and} \quad x+19 = 64.$$

Art. 333.

1.
$$3x^3+42x^{\frac32}=3321.$$
$$x^3+14x^{\frac32}=1107.$$
Completing the square, $$x^3+14x^{\frac32}+49=1156.$$
Extracting square root, $$x^{\frac32}+7=\pm 34.$$
$$x^{\frac32}=27 \text{ or } -41.$$
$$x^{\frac12}=3 \text{ or } \sqrt[3]{-41}.$$
$$x=9 \text{ or } \sqrt[3]{1681}.$$

2.
$$x^{\frac43}+7x^{\frac23}=44.$$
Multiplying by 4 and adding 7^2,
$$4x^{\frac43}+28x^{\frac23}+49=225.$$
Extracting square root, $$2x^{\frac23}+7=\pm 15.$$
Whence, $$x^{\frac23}=4 \text{ or } -11,$$
$$x^{\frac13}=\pm 2 \text{ or } \pm\sqrt{-11},$$
and $$x=\pm 8 \text{ or } \pm\sqrt{11^3}.$$

3.
$$4x^{\frac13}+x^{\frac16}=39.$$
Completing the square, $$4x^{\frac13}+x^{\frac16}+\tfrac{1}{16}=\tfrac{625}{16}.$$
Extracting square root, $$2x^{\frac16}+\tfrac14=\pm\tfrac{25}{4}.$$
Whence, $$x^{\frac16}=3 \text{ or } -3\tfrac14,$$
and $$x=729 \text{ or } (-\tfrac{13}{4})^6.$$

4.
$$3x^6+42x^3=3321.$$
$$x^6+14x^3=1107.$$
Completing the square, $$x^6+14x^3+49=1156.$$
Extracting the square root, $$x^3+7=\pm 34.$$
Whence, $$x^3=27 \text{ or } -41,$$
and $$x=3 \text{ or } \sqrt[3]{-41}.$$

5. $$2x^3-17x^{\frac{3}{2}} = -8.$$

Multiplying by 2 and adding $(\frac{17}{2})^2$,

$$4x^3-34x^{\frac{3}{2}}+\frac{289}{4} = \frac{225}{4}.$$

Extracting square root, $$2x^{\frac{3}{2}}-\frac{17}{2} = \pm\frac{15}{2}.$$

Whence, $$x^3 = 8 \text{ or } \tfrac{1}{2},$$

and $$x = 2 \text{ or } \tfrac{1}{2}\sqrt[3]{2}.$$

6. $$3x^{\frac{4}{3}}-4x^{\frac{2}{3}} = 7.$$

$$x^{\frac{4}{3}}-\tfrac{4}{3}x^{\frac{2}{3}} = \tfrac{7}{3}.$$

Completing the square, $$x^{\frac{4}{3}}-\tfrac{4}{3}x^{\frac{2}{3}}+\tfrac{4}{9} = \tfrac{25}{9}.$$

Extracting the square root, $$x^{\frac{2}{3}}+\tfrac{2}{3} = \pm\tfrac{5}{3}.$$

Whence, $$x^{\frac{2}{3}} = 1 \text{ or } -\tfrac{7}{3},$$

$$x^{\frac{1}{3}} = 1 \text{ or } \sqrt[3]{-\tfrac{7}{3}},$$

$$x = 1 \text{ or } \tfrac{1}{3}\sqrt[3]{63}.$$

7. $$x^{10}+31x^5 = 32.$$

Completing the square, $$x^{10}+31x^5+\tfrac{961}{4} = \tfrac{1089}{4}.$$

Extracting square root, $$x^5+\tfrac{31}{2} = \pm\tfrac{33}{2}.$$

Whence, $$x^5 = 1 \text{ or } -32,$$

and $$x = 1 \text{ or } -2.$$

8. $$3x+2x^{\frac{1}{2}} = 1.$$

$$x+\tfrac{2}{3}x^{\frac{1}{2}} = \tfrac{1}{3}.$$

Completing the square, $$x+\tfrac{2}{3}x^{\frac{1}{2}}+\tfrac{1}{9} = \tfrac{4}{9}.$$

Extracting square root, $$x^{\frac{1}{2}}+\tfrac{1}{3} = \pm\tfrac{2}{3}.$$

Whence, $$x^{\frac{1}{2}} = +\tfrac{1}{3} \text{ or } -1,$$

and $$x = \tfrac{1}{9} \text{ or } 1.$$

9. $$x^4+4x^2 = 12.$$

Completing the square, $$x^4+4x^2+4 = 16.$$

Extracting the square root. $$x^2+2 = \pm 4.$$

Whence, $$x^2 = 2 \text{ or } -6,$$

$$x = \pm\sqrt{2} \text{ or } \pm\sqrt{-6}.$$

10. $$5x^{\frac{1}{2}}+x^{\frac{1}{4}}=22.$$

$$x^{\frac{1}{2}}+\tfrac{1}{5}x^{\frac{1}{4}}=\tfrac{22}{5}.$$

Completing the square, $$x^{\frac{1}{2}}+\tfrac{1}{5}x^{\frac{1}{4}}+\tfrac{1}{100}=\tfrac{441}{100}.$$

Extracting the square root, $$x^{\frac{1}{4}}+\tfrac{1}{10}=\pm\tfrac{21}{10}.$$

Whence, $$x^{\frac{1}{4}}=2 \text{ or } -\tfrac{22}{10},$$

$$x=16 \text{ or } (-\tfrac{11}{5})^4.$$

11. $$x^3-x^{\frac{3}{2}}=56.$$

Completing the square, $$x^3-x^{\frac{3}{2}}+\tfrac{1}{4}=\tfrac{225}{4}.$$

Extracting the square root, $$x^{\frac{3}{2}}-\tfrac{1}{2}=\pm\tfrac{15}{2}.$$

Whence, $$x^{\frac{3}{2}}=8 \text{ or } -7,$$

$$x^{\frac{1}{2}}=2 \text{ or } \sqrt[3]{-7},$$

$$x=4 \text{ or } \sqrt[3]{49}.$$

12. $$x^{\frac{6}{5}}+x^{\frac{3}{5}}=756.$$

Completing the square, $$x^{\frac{6}{5}}+x^{\frac{3}{5}}+\tfrac{1}{4}=\tfrac{3025}{4}.$$

Whence, $$x^{\frac{3}{5}}+\tfrac{1}{2}=\pm\tfrac{55}{2},$$

$$x^{\frac{3}{5}}=27 \text{ or } -28,$$

$$x^{\frac{1}{5}}=3 \text{ or } \sqrt[3]{-28},$$

$$x=243 \text{ or } \sqrt[3]{-28^5}$$

13. $$2x^{\frac{2}{3}}-6x^{\frac{1}{3}}=20.$$

$$x^{\frac{2}{3}}-3x^{\frac{1}{3}}=10.$$

Completing the square, $$x^{\frac{2}{3}}-3x^{\frac{1}{3}}+\tfrac{9}{4}=\tfrac{49}{4}.$$

Extracting the square root, $$x^{\frac{1}{3}}-\tfrac{3}{2}=\pm\tfrac{7}{2}.$$

Whence, $$x^{\frac{1}{3}}=5 \text{ or } -2,$$

$$x=125 \text{ or } -8.$$

14. $$x^{\frac{1}{3}}+\sqrt{2}=\frac{2}{x^{\frac{1}{3}}-\sqrt{2}}.$$

Clearing of fractions, $$x^{\frac{2}{3}}-2=2.$$

Whence, $$x^{\frac{2}{3}}=4.$$

$$x^{\frac{1}{3}}=\pm 2, \qquad x=\pm 8.$$

15.
$$3x^{\frac{4}{5}}+7x^{\frac{2}{5}} = 76.$$
$$x^{\frac{4}{5}}+\tfrac{7}{3}x^{\frac{2}{5}} = \tfrac{76}{3}.$$

Completing the square, $x^{\frac{4}{5}}+\frac{7}{3}x^{\frac{2}{5}}+\frac{49}{36} = \frac{961}{36}.$

Extracting the square root, $x^{\frac{2}{5}}+\frac{7}{6} = \pm\frac{31}{6}.$

Whence,
$$x^{\frac{2}{5}} = 4 \text{ or } -\tfrac{19}{3},$$
$$x^{\frac{1}{5}} = 2 \text{ or } \pm\tfrac{1}{3}\sqrt{-57},$$
$$x = 32 \text{ or } (\pm\tfrac{1}{3}\sqrt{57})^5.$$

16.
$$x^2-2x+5+6(x^2-2x+5)^{\frac{1}{2}} = 6.$$

Let $x^2-2x+5 = y.$ (1)

Then
$$y+6y^{\frac{1}{2}} = 6.$$
$$y^{\frac{1}{2}}+3 = \pm\sqrt{15},$$
$$y^{\frac{1}{2}} = -3+\sqrt{15}, \text{ or } -3-\sqrt{15}.$$
$$y = 24-6\sqrt{15}, \text{ or } 4.$$

Substituting value of y in (1),
$$x^2-2x+5 = 24-6\sqrt{15}.$$

Whence,
$$x^2-2x = 19-6\sqrt{15},$$
$$x-1 = \sqrt{20-6\sqrt{15}},$$

and
$$x = 1\pm 2\sqrt{15}.$$

Substituting second value of y in (1),
$$x^2-2x+5 = 4,$$
$$x-1 = 0,$$
$$x = 1.$$

17.
$$x+16-3\sqrt{x+16} = 10.$$

Let $x+16 = y.$ (1)

Then
$$y-3y^{\frac{1}{2}} = 10.$$

Whence,
$$y^{\frac{1}{2}}-\tfrac{3}{2} = \pm\tfrac{7}{2},$$
$$y^{\frac{1}{2}} = 5 \text{ or } -2,$$

and
$$y = 25 \text{ or } 4.$$

Substituting values of y in (1), $x+16 = 25$, or $x+16 = 4.$

Whence,
$$x = 9 \text{ or } -12.$$

18. $$(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}} = 6.$$

Let $$(x+12)^{\frac{1}{2}} = y. \quad (1)$$

Then $$y+y^{\frac{1}{2}} = 6.$$

Whence, $$y^{\frac{1}{2}}+\tfrac{1}{2} = \pm\tfrac{5}{2},$$

$$y^{\frac{1}{2}} = 2 \text{ or } -3,$$

and $$y = 4 \text{ or } 9.$$

Substituting values of y in (1), $(x+12)^{\frac{1}{2}} = 4$, and $(x+12)^{\frac{1}{2}} = 9$.

Whence, $$x = 4, \quad\text{and}\quad x = 69.$$

19. $$(2x^2-7x+6)+2(2x^2-7x+6)^{\frac{1}{2}} = 0.$$

Let $$2x^2-7x+6 = y. \quad (1)$$

Then $$y+2y^{\frac{1}{2}} = 0.$$

Whence, $$y^{\frac{1}{2}}+1 = \pm 1,$$

and $$y = 4 \text{ or } 0.$$

Substituting first value of y in (1),

$$2x^2-7x+6 = 0.$$

$$2x^2-7x = -6.$$

Whence, $$x^2-\tfrac{7}{2}x = -3,$$

$$x-\tfrac{7}{4} = \pm\tfrac{1}{4}, \qquad x = 2 \text{ or } +1\tfrac{1}{2}.$$

Substituting second value of y in (1),

$$2x^2-7x+6 = 4.$$

$$2x^2-7x = -2.$$

$$x^2-\tfrac{7}{2}x = -1.$$

$$x-\tfrac{7}{4} = \pm\tfrac{1}{4}\sqrt{33}.$$

$$x = \tfrac{1}{4}(7 \pm \sqrt{33}).$$

20. $$(x^2-4x+4)^2+5(x^2-4x+4) = 126.$$

Let $$x^2-4x+4 = y. \quad (1)$$

Then $$y^2+5y = 126,$$

and $$y = 9 \text{ or } -14.$$

Substituting second value of y in (1),

$$x^2-4x+4 = 9.$$

Then $x^2-4x = -18,$

and $x = 2 \pm \sqrt{14}.$

Substituting first value of y in (1),

$$x^2-4x+4 = 9,$$

$$x-2 = \pm 3, \qquad x = 5 \text{ or } -1.$$

21. $$\left(x+\frac{1}{x}\right)^2 + 2\left(x+\frac{1}{x}\right) = \frac{160}{9}.$$

Let $x + \frac{1}{x} = y.$ (1)

Then $y^2+2y = \frac{160}{9},$

and $y = -\frac{16}{3} \text{ or } \frac{10}{3}.$

Substituting first value of y in (1),

$$x + \frac{1}{x} = -\frac{16}{3}.$$

Whence, $x^2+\frac{16}{3}x = -1,$

and $x = \frac{1}{3}(-8 \pm \sqrt{55}).$

Substituting second value of y in (1),

$$x + \frac{1}{x} = \frac{10}{3}.$$

$$3x^2+3 = 10x. \qquad 3x^2-10x = -3.$$

$$9x^2-30x = -9.$$

$$36x^2-120x = -36.$$

$$36x^2-120x+100 = 64.$$

$$6x-10 = \pm 8. \qquad x = \frac{1}{3} \text{ or } 3.$$

22. $$x+21+(x+21)^{\frac{1}{2}} = 12.$$

Let $x+21 = y.$ (1)

$$y+y^{\frac{1}{2}} = 12.$$

$$y^{\frac{1}{2}} = -4 \text{ or } 3,$$

$$y = 16 \text{ or } 9.$$

Substituting first value of y in (1),

$$x+21 = 16. \quad \text{Whence,} \quad x = -5.$$

Substituting second value of y in (1),

$$x+21 = 9. \quad \text{Whence,} \quad x = -12.$$

23. $$(2x+6)^{\frac{1}{2}}+(2x+6)^{\frac{1}{4}} = 6.$$

Let $$2x+6 = y. \qquad (1)$$

Then $$y^{\frac{1}{2}}+y^{\frac{1}{4}} = 6.$$

Whence, $$y^{\frac{1}{4}} = -3 \text{ or } 2,$$

and $$y = 81 \text{ or } 16.$$

Substituting first value of y in (1),

$$2x+6 = 81.$$

Whence, $$x = 37\tfrac{1}{2}.$$

Substituting second value of y in (1),

$$2x+6 = 16. \qquad \text{Whence,} \quad x = 5.$$

24. $$x+6+\sqrt{x+6} = 8+3\sqrt{x+6}.$$

$$x+6-2\sqrt{x+6} = 8.$$

Let $$x+6 = y. \qquad (1)$$

$$y-2y^{\frac{1}{2}} = 8.$$

Whence, $$y = 16 \text{ or } 4.$$

Substituting values of y in (1), $x+6 = 16$, and $x+6 = 4$.

Whence, $x = 10$, and $x = -2$.

25. $$x+5 = \sqrt{x+5}+6.$$

$$x+5-\sqrt{x+5} = 6.$$

Let $$x+5 = y. \qquad (1)$$

Then $$y-y^{\frac{1}{2}} = 6.$$

Whence, $$y = 9 \text{ or } 4.$$

Substituting values of y in (1), $x+5 = 9$, and $x+5 = 4$.

Whence, $x = 4$, and $x = -1$.

Art. 340.

1. $5x-y = 13$; (1) $\qquad 3xy-2y^2 = 10.$ (2)

From (1), $$x = \frac{13+y}{5}.$$

Substituting in (2), $$3\left(\frac{13+y}{5}\right)y-2y^2 = 10.$$

Clearing of fractions, $7y^2-39y = -50.$

Completing square and extracting square root,

$$7y-\tfrac{39}{2} = \pm\tfrac{11}{2}.$$

Whence, $y = 3\tfrac{4}{7}$ or $2.$

Hence, $x = 3\tfrac{11}{35}$ or $3.$

2. $5x+2y = 7;$ (1) $7x^2-8xy = 159.$ (2)

From (1), $y = \dfrac{7-5x}{2}.$

Substituting in (2), $7x^2-8x\left(\dfrac{7-5x}{2}\right) = 159.$

Clearing of fractions, $27x^2-28x = 159.$

Completing square and extracting square root,

$$27x-14 = \pm 67.$$

Whence, $x = 3$ or $-\tfrac{53}{27}.$

Hence, $y = -4$ or $\tfrac{227}{27}.$

3. $x+4y = 14;$ (1) $4x-2y+y^2 = 11.$ (2)

From (1), $x = 14-4y.$

Substituting in (2), $4(14-4y)-2y+y^2 = 11.$

Transposing and uniting, $y^2-18y = -45.$

Completing square and extracting square root,

$$y-9 = \pm 6.$$

Whence, $y = 3$ or $15.$

Hence, $x = 2$ or $-46.$

4. $x-y = 12;$ (1) $x^2+y^2 = 74.$ (2)

From (1), $x = 12+y.$

Substituting in (2), $(12+y)^2+y^2 = 74.$

Transposing and uniting, $2y^2+24y = -70,$

or, $y^2+12y = -35.$

Completing square and extracting square root,

$$y+6 = \pm 1.$$

Whence, $y = -5$ or $-7.$

Hence, $x = 7$ or $5.$

5. $$2x+y=22; \quad (1) \qquad xy+2y^2=120. \quad (2)$$

From (1), $$x=\frac{22-y}{2}.$$

Substituting in (2), $$y\left(\frac{22-y}{2}\right)+2y^2=120.$$

Simplifying, $$3y^2+22y=240.$$

Completing square and extracting square root,

$$3y+11=\pm 29.$$

Whence, $$y=6 \text{ or } -13\tfrac{1}{3}.$$

Hence, $$x=8 \text{ or } 17\tfrac{2}{3}.$$

6. $$10x+y=3xy; \quad (1) \qquad y-x=2. \quad (2)$$

From (2), $$y=2+x.$$

Substituting in (1), $$10x+2+x=3x(2+x).$$

Simplifying, $$3x^2-5x=2.$$

Completing square and extracting square root,

$$6x-5=\pm 7.$$

Whence, $$x=2 \text{ or } -\tfrac{1}{3}.$$

Hence, $$y=4 \text{ or } 1\tfrac{2}{3}.$$

7. $$2x-3y=1; \quad (1) \qquad 2x^2+xy-5y^2=20. \quad (2)$$

From (1), $$x=\frac{1+3y}{2}.$$

Substituting in (2),

$$2\left(\frac{1+3y}{2}\right)^2+\left(\frac{1+3y}{2}\right)y-5y^2=20.$$

Simplifying, $$4y^2+14y=78.$$

Completing square and extracting square root,

$$2y+\tfrac{7}{2}=\pm\tfrac{19}{2}.$$

Whence, $$y=3 \text{ or } -6\tfrac{1}{2}.$$

Hence, $$x=5 \text{ or } -9\tfrac{1}{4}.$$

8. $$3x+y=18; \quad (1) \qquad x^2+2y^2=43. \quad (2)$$

From (1), $$y=18-3x.$$

Substituting in (2), $$x^2+2(18-3x)^2=43.$$

Simplifying, $$19x^2-216x=-605.$$

Completing square and extracting square root,

$$19x-108 = \pm 13.$$

Whence, $x = 6\tfrac{7}{19}$ or 5.

Hence, $y = -1\tfrac{3}{19}$ or 3.

9. $x+y = 8$; (1) $\quad xy = 2x+y+2$. (2)

From (1), $y = 8-x$.

Substituting in (2), $x(8-x) = 2x+8-x+2$.

Transposing and uniting, $x^2-7x = -10$.

Completing square and extracting square root,

$$x-\tfrac{7}{2} = \pm \tfrac{3}{2}.$$

Whence, $x = 5$ or 2.

Hence, $y = 3$ or 6.

10. $4(x+y) = 5$; (1) $\quad 8xy = 3$. (2)

From (2), $x = \dfrac{3}{8y}$.

Substituting in (1), $\dfrac{12}{8y} + 4y = 5$.

Simplifying, $8y^2-10y = -3$,

or, $4y^2-5y = -\tfrac{3}{2}$.

Completing square and extracting square root,

$$2y-\tfrac{5}{4} = \pm \tfrac{1}{4}.$$

Whence, $y = \tfrac{3}{4}$ or $\tfrac{1}{2}$.

Hence, $x = \tfrac{1}{2}$ or $\tfrac{3}{4}$.

11. $4x = 9y$; (1) $\quad x+y^2 = 25$. (2)

From (1), $x = \tfrac{9}{4}y$.

Substituting in (2), $\tfrac{9}{4}y+y^2 = 25$.

Completing square and extracting square root,

$$y+\tfrac{9}{8} = \pm \tfrac{41}{8}.$$

Whence, $y = 4$ or $-6\tfrac{1}{4}$.

Hence, $x = 9$ or $-14\tfrac{1}{16}$.

12. $2x - 3y = 1; \quad (1) \qquad 8x^2 - 22xy + 15y^2 = 5. \quad (2)$

From (1), $$x = \frac{1+3y}{2}.$$

Substituting in (2),

$$8\left(\frac{1+3y}{2}\right)^2 - 22y\left(\frac{1+3y}{2}\right) + 15y^2 = 5.$$

Clearing of fractions,

$$2 + 12y + 18y^2 - 11y - 33y^2 + 15y^2 = 5.$$

Transposing and uniting, $$y = 3.$$

Hence, $$x = 5.$$

13. $x + 3y = 7; \quad (1) \qquad x^2 + 3xy + y^2 = 11. \quad (2)$

From (1), $$x = 7 - 3y.$$

Substituting in (2),

$$(7-3y)^2 + 3y(7-3y) + y^2 = 11.$$

Simplifying, $$y^2 - 21y = -38.$$

Completing square and extracting square root,

$$y - \tfrac{21}{2} = \pm \tfrac{17}{2}.$$

Whence, $$y = 19 \text{ or } 2.$$

Hence, $$x = -50 \text{ or } 1.$$

14. $2x - 3y = 2; \quad (1) \qquad 4x^2 + 9y^2 = 100. \quad (2)$

From (1), $$x = \frac{2+3y}{2}.$$

Substituting in (2), $$4\left(\frac{2+3y}{2}\right)^2 + 9y^2 = 100.$$

Simplifying, $$3y^2 + 2y = 16.$$

Multiplying by 12, $$36y^2 + 24y = 192.$$

Adding 2^2 and extracting square root,

$$6y + 2 = \pm 14.$$

Whence, $$y = 2 \text{ or } -2\tfrac{2}{3}.$$

Hence, $$x = 4 \text{ or } -3.$$

15. $x-y=3$; (1) $2x^2+3y^2-xy=31$. (2)

From (1), $y=x-3$.

Substituting in (2),

$$2x^2+3(x-3)^2-x(x-3)=31.$$

Simplifying, $4x^2-15x=4$,

or, $x^2-\frac{15}{4}x=1$.

Completing square and extracting square root,

$$x-\tfrac{15}{8}=\pm\tfrac{17}{8}.$$

Whence, $x=4$ or $-\frac{1}{4}$.

Hence, $y=1$ or $-3\frac{1}{4}$.

16. $2x+3y=16$; (1) $4x^2-xy=90$. (2)

From (1), $x=\dfrac{16-3y}{2}$.

Substituting in (2),

$$4\left(\frac{16-3y}{2}\right)^2-y\left(\frac{16-3y}{2}\right)=90.$$

Simplifying, $21y^2-208y=-332$.

Completing square and extracting square root,

$$21y-104=\pm 62.$$

Whence, $y=2$ or $7\frac{19}{21}$.

Hence, $x=5$ or $-3\frac{6}{7}$.

17. $2x+11y=15$; (1) $4x^2+121y^2=137$. (2)

From (1), $x=\dfrac{15-11y}{2}$.

Substituting in (2),

$$4\left(\frac{15-11y}{2}\right)^2+121y^2=137.$$

Simplifying, $242y^2-330y=-88$,

or, $11y^2-15y=-4$.

Completing the square and extracting square root,

$$22y-15=\pm 7.$$

Whence, $y=1$ or $\frac{4}{11}$.

Hence, $x=2$ or $5\frac{1}{2}$.

18. $x+y=20$; (1) $x^2-2y^2=71$. (2)

From (1), $x=20-y$.

Substituting in (2), $(20-y)^2-2y^2=71$.

Simplifying, $y^2+40y=329$.

Completing square and extracting square root,

$$y+20=\pm 27.$$

Whence, $y=7$ or -47.

Hence, $x=13$ or 67.

19. $x+y=7$; (1) $x^2+2y^2=34$. (2)

From (1), $x=7-y$.

Substituting in (2), $(7-y)^2+2y^2=34$.

Simplifying, $3y^2-14y=-15$.

Completing square and extracting square root,

$$3y-7=\pm 2.$$

Whence, $y=3$ or $1\frac{2}{3}$.

Hence, $x=4$ or $5\frac{1}{3}$.

20. $x+y=2$; (1) $x^2-2xy-y^2=1$.

From (1), $x=2-y$.

Substituting in (2),

$$(2-y)^2-2y(2-y)-y^2=1.$$

Simplifying, $2y^2-8y=-8$.

Completing square and extracting square root,

$$2y-4=\sqrt{10}.$$

Whence, $y=2\pm\frac{1}{2}\sqrt{10}$.

Hence, $x=\mp\frac{1}{2}\sqrt{10}$.

Art. 342.

1. $5x^2+3xy=26$; (1) $3y^2+2xy=7$. (2)

Let $x=ry$.

Then $\begin{cases}5x^2+3xy=26\\3y^2+2xy=7\end{cases}=\begin{cases}5r^2y^2+3ry^2=26, & (3)\\3y^2+2ry^2=7. & (4)\end{cases}$

From (3), $$y^2=\frac{26}{5r^2+3r}. \qquad (5)$$

From (4), $$y^2 = \frac{7}{3+2r}. \quad (6)$$

Hence, $$\frac{26}{5r^2+3r} = \frac{7}{3+2r},$$

and $$78+52r = 35r^2+21r.$$

Transposing and uniting,

$$35r^2-31r = 78.$$

Completing square and extracting square root,

$$35r-\tfrac{31}{2} = \pm\tfrac{109}{2}.$$

Whence, $$r = 2 \text{ or } -1\tfrac{4}{35}.$$

Substituting values of r in (6),

$$y^2 = \frac{7}{3+4}, \quad \text{or} \quad y^2 = \frac{7}{3+2\left(-\frac{39}{35}\right)}.$$

Whence, $$y = \pm 1 \text{ or } \pm\tfrac{7}{9}\sqrt{15}.$$

Hence, $$x = \pm 2 \text{ or } \pm\tfrac{13}{15}\sqrt{15}.$$

2. $x^2+xy = 12$; (1) $xy-2y^2 = 1$. (2)

Let $$x = ry.$$

Then $$\left.\begin{array}{r} x^2+xy = 12 \\ xy-2y^2 = 1 \end{array}\right\} = \left\{\begin{array}{r} r^2y^2+ry^2 = 12, \quad (3) \\ ry^2-2y^2 = 1. \quad (4) \end{array}\right.$$

From (3), $$y^2 = \frac{12}{r^2+r}. \quad (5)$$

From (4), $$y^2 = \frac{1}{r-2}. \quad (6)$$

Hence, $$\frac{12}{r^2+r} = \frac{1}{r-2}.$$

Simplifying, $$12r-24 = r^2+r, \quad \text{or} \quad r^2-11r = -24.$$

Completing square and extracting square root,

$$r-\tfrac{11}{2} = \pm\tfrac{5}{2}.$$

Whence, $$r = 8 \text{ or } 3.$$

Substituting values of r in (6),

$$y^2 = \tfrac{1}{6}, \quad \text{or} \quad y^2 = 1.$$

Whence, $$y = \pm\tfrac{1}{6}\sqrt{6}, \quad \text{or} \quad y = \pm 1.$$

Hence, $$x = \pm\tfrac{4}{3}\sqrt{6} \text{ or } \pm 3.$$

3. $5x^2+6y^2-12xy = 2;$ (1) $4x^2+5y^2-10xy = 1.$ (2)

Let $x = ry.$

Then $\left.\begin{array}{l} 5x^2+6y^2-12xy = 2 \\ 4x^2+5y^2-10xy = 1 \end{array}\right\} = \left\{\begin{array}{l} 5r^2y^2+6y^2-12ry^2 = 2, \\ 4r^2y^2+5y^2-10ry^2 = 1. \end{array}\right\}$ (3) (4)

From (3), $$y^2 = \frac{2}{5r^2+6-12r}. \quad (5)$$

From (4), $$y^2 = \frac{1}{4r^2+5-10r}. \quad (6)$$

Hence, $$\frac{2}{5r^2+6-12r} = \frac{1}{4r^2+5-10r}.$$

Simplifying, $8r^2+10-20r = 5r^2+6-12r,$

or, $3r^2-8r = -4.$

Completing square and extracting square root,

$$3r-4 = \pm 2.$$

Whence, $r = 2$ or $\frac{2}{3}$.

Substituting values of r in (6),

$$y^2 = \frac{1}{16+5-20},$$

$$y^2 = \frac{1}{4(\frac{2}{3})^2+5-10(\frac{2}{3})}$$

Whence, $y = \pm 1$ or $\pm 3.$

Hence, $x = \pm 2.$

4. $4y^2 = 3x^2+xy;$ (1) $x^2+y^2 = 25.$ (2)

From (2), $y^2 = 25-x^2.$

Substituting in (1), $4(25-x^2) = 3x^2+xy,$

or $$7x^2+xy = 100. \quad (3)$$

Let $x = ry.$

Then $\left.\begin{array}{l} x^2+y^2 = 25, \ (2) \\ 7x^2+xy = 100, \ (3) \end{array}\right\} = \left\{\begin{array}{l} r^2y^2+y^2 = 25, \\ 7r^2y^2+ry^2 = 100. \end{array}\right\}$ (4) (5)

From (4), $$y^2 = \frac{25}{r^2+1}. \quad (6)$$

From (5), $$y^2 = \frac{100}{7r^2+r}. \quad (7)$$

Hence, $$\frac{25}{r^2+1} = \frac{100}{7r^2+r}.$$

Simplifying, $$75r^2+25r = 100,$$

or, $$3r^2+r = 4.$$

Completing square and extracting square root,

$$6r+1 = \pm 7.$$

Hence, $$r = -\tfrac{4}{3} \text{ or } 1.$$

Substituting in (6), $$y^2 = \frac{25}{(-\frac{4}{3})^2+1} \text{ or } \frac{25}{1+1}.$$

Whence, $$y = \mp 3 \text{ or } \pm \frac{5}{\sqrt{2}}.$$

Hence $$x = \pm 4 \text{ or } \pm \frac{5}{\sqrt{2}}.$$

5. $x^2-2xy-y^2 = 31$; (1) $x^2+4xy-2y^2 = 202.$ (2)

Let $$x = ry.$$

Then $$\left\{\begin{matrix} x^2-2xy-y^2 = 31 \\ x^2+4xy-2y^2 = 202 \end{matrix}\right\} = \left\{\begin{matrix} r^2y^2-2ry^2-y^2 = 31, & (3) \\ r^2y^2+4ry^2-2y^2 = 202. & (4) \end{matrix}\right.$$

From (3), $$y^2 = \frac{31}{r^2-2r-1}. \qquad (5)$$

From (4), $$y^2 = \frac{202}{r^2+4r-2}. \qquad (6)$$

Hence, $$\frac{31}{r^2-2r-1} = \frac{202}{r^2+4r-2}.$$

Simplifying, $$171r^2-528r = 140.$$

Completing square and extracting square root,

$$171r-264 = \pm 306.$$

$$r = 3\tfrac{1}{3} \text{ or } -\tfrac{14}{57}.$$

Substituting values of r in (5),

$$y^2 = \frac{202}{\frac{100}{9}+\frac{40}{3}-2} \text{ or } \frac{202}{(-\frac{14}{57})^2+4(\frac{14}{57})-2}.$$

Whence, $$y = \pm 3 \text{ or } \pm \tfrac{57}{47}\sqrt{-47}.$$

Hence, $$x = \pm 10 \text{ or } \mp \tfrac{14}{47}\sqrt{-47}.$$

6. $$10(x^2+y^2) = 29xy; \quad (1) \qquad\qquad xy = 10. \quad (2)$$

Substituting value of xy in (1),

$$x^2+y^2 = 29. \tag{3}$$

Let $$x = ry.$$

Then $$\left\{\begin{aligned} x^2+y^2 &= 29 \\ xy &= 10 \end{aligned}\right\} = \left\{\begin{aligned} r^2y^2+y^2 &= 29, \quad (4) \\ ry^2 &= 10. \quad (5) \end{aligned}\right\}$$

From (4), $$y^2 = \frac{29}{r^2+1}. \tag{6}$$

From (5), $$y^2 = \frac{10}{r}. \tag{7}$$

Hence, $$\frac{29}{r^2+1} = \frac{10}{r}.$$

Simplifying, $$r^2-\tfrac{29}{10} = -1.$$

Completing square and extracting square root,

$$r-\tfrac{29}{20} = \tfrac{21}{20}.$$

Whence, $$r = \tfrac{5}{2}.$$

Substituting in (7), $$y^2 = 4.$$

Whence, $$y = \pm 2. \qquad \text{Hence,} \quad x = \pm 5.$$

7. $$4x^2-2xy+3y^2 = 45; \quad (1) \qquad\qquad xy+y^2 = -2. \quad (2)$$

Let $$x = ry.$$

Then $$\left\{\begin{aligned} 4x^2-2xy+3y^2 &= 45 \\ xy+y^2 &= -2 \end{aligned}\right\} = \left\{\begin{aligned} 4r^2y^2-2ry^2+3y^2 &= 45, \quad (3) \\ ry^2+y^2 &= -2. \quad (4) \end{aligned}\right\}$$

From (3), $$y^2 = \frac{45}{4r^2-2r+3}. \tag{5}$$

From (4), $$y^2 = \frac{-2}{r+1}. \tag{6}$$

Whence, $$\frac{45}{4r^2-2r+3} = \frac{-2}{r+1}.$$

Simplifying, $$8r^2+41r = -51.$$

Completing square and extracting square root,

$$16r+41 = \pm 7.$$

Whence, $$r = -2\tfrac{1}{8} \text{ or } -3.$$

Substituting in (6), $y^2 = \dfrac{-2}{-2}$, or $\dfrac{-2}{-2\frac{1}{8}+1}$.

Whence, $y = \pm 1$ or $\pm 1\frac{1}{3}$.

Hence, $x = \mp 3$ or $\mp 2\frac{5}{6}$.

8. $x^2+xy = 15$; (1) $xy-y^2 = 2$. (2)

Let $x = ry$.

Then $\begin{Bmatrix} x^2+xy = 15 \\ xy-y^2 = 2 \end{Bmatrix} = \begin{Bmatrix} r^2y^2+ry^2 = 15, & (3) \\ ry^2-y^2 = 2. & (4) \end{Bmatrix}$

From (3), $y^2 = \dfrac{15}{r^2+r}$. (5)

From (4), $y^2 = \dfrac{2}{r-1}$. (6)

Whence, $\dfrac{15}{r^2+r} = \dfrac{2}{r-1}$.

Simplifying, $2r^2-13r = -15$.

Completing square and extracting square root,

$$4r-13 = \pm 7.$$

Whence, $r = 5$ or $\frac{3}{2}$.

Substituting in (6), $y^2 = \dfrac{2}{5-1}$ or $\dfrac{2}{\frac{3}{2}-1}$.

Whence, $y = \pm 2$ or $\pm \frac{1}{2}\sqrt{2}$.

Hence, $x = \pm 3$ or $\pm \frac{5}{2}\sqrt{2}$.

9. $x^2+4y^2+4xy = 256$; (1) $3y^2-x^2 = 39$. (2)

Let $x = ry$.

Then $\begin{Bmatrix} x^2+4y^2+4xy = 256 \\ 3y^2-x^2 = 39 \end{Bmatrix} = \begin{Bmatrix} r^2y^2+4y^2+4ry^2 = 256, & (3) \\ 3y^2-r^2y^2 = 39. & (4) \end{Bmatrix}$

From (3), $y^2 = \dfrac{256}{r^2+4+4r}$. (5)

From (4), $y^2 = \dfrac{39}{3-r^2}$. (6)

Hence, $\dfrac{256}{r^2+4+4r} = \dfrac{39}{3-r^2}$.

Simplifying, $295r^2 + 156r = 612.$

Completing square and extracting square root,

$$295r + 78 = \pm 432.$$

Whence, $r = -1\tfrac{43}{59}$ or $1\tfrac{1}{5}$.

Substituting in (6), $y^2 = \dfrac{39}{3-(\frac{6}{5})^2}$ or $\dfrac{39}{3-(\frac{102}{59})^2}.$

Whence, $y = \pm 5$ or $\mp 59.$

Hence, $x = \pm 6$ or $\mp 102.$

10. $2x^2 + 3xy + y^2 = 20;$ (1) $5x^2 + 4y^2 = 41.$ (2)

Let $x = ry.$

Then $\left\{\begin{aligned} 2x^2 + 3xy + y^2 &= 20 \\ 5x^2 + 4y^2 &= 41 \end{aligned}\right\} = \left\{\begin{aligned} 2r^2y^2 + 3ry^2 + y^2 &= 20, \\ 5r^2y^2 + 4y^2 &= 41. \end{aligned}\right\}$ (3) (4)

From (3), $y^2 = \dfrac{20}{2r^2 + 3r + 1}.$ (5)

From (4), $y^2 = \dfrac{41}{5r^2 + 4}.$ (6)

Whence, $\dfrac{20}{2r^2 + 3r + 1} = \dfrac{41}{5r^2 + 4}.$

Simplifying, $18r^2 - 123r = -39,$

or, $6r^2 - 41r = -13.$

Completing square and extracting square root,

$$12r - 41 = \pm 37.$$

$$r = \tfrac{1}{3} \text{ or } 6\tfrac{1}{2}.$$

Substituting in (6), $y^2 = \dfrac{41}{5\,(\frac{1}{3})^2 + 4}$ or $\dfrac{41}{5\,(6\frac{1}{2})^2 + 4}.$

Whence, $y = \pm 3$ or $\pm \tfrac{2}{21}\sqrt{21}.$

Hence, $x = \pm 1$ or $\pm \tfrac{13}{21}\sqrt{21}.$

11. $5x^2 - 3xy + 2y^2 = 35;$ (1) $2x^2 + 4xy - y^2 = 38.$ (2)

Let $x = ry.$

Then, $\left\{\begin{aligned} 5x^2 - 3xy + 2y^2 &= 35 \\ 2x^2 + 4xy - y^2 &= 38 \end{aligned}\right\} = \left\{\begin{aligned} 5r^2y^2 - 3ry^2 + 2y^2 &= 35, \\ 2r^2y^2 + 4ry^2 - y^2 &= 38. \end{aligned}\right\}$ (3) (4)

From (3), $$y^2 = \frac{35}{5r^2-3r+2}. \quad (5)$$

From (4), $$y^2 = \frac{38}{2r^2+4r-1}. \quad (6)$$

Hence, $$\frac{35}{5r^2-3r+2} = \frac{38}{2r^2+4r-1}.$$

Simplifying, $$120r^2-254r = -111.$$

Completing square and extracting square root,

$$120r-127 = \pm 53.$$

$$r = \tfrac{37}{60} \text{ or } 1\tfrac{1}{2}.$$

Substituting in (6), $$y^2 = \frac{38}{2\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)-1} \text{ or } \frac{38}{2\left(\frac{37}{60}\right)^2+4\left(\frac{37}{60}\right)-1}.$$

Whence, $$y = \pm 2 \text{ or } \pm\frac{60}{\sqrt{211}}.$$

Hence, $$x = \pm 3 \text{ or } \pm\frac{37}{\sqrt{211}}.$$

12. $$x^2-xy = 6; \quad (1) \qquad x^2+y^2 = 61. \quad (2)$$

Let $$x = ry.$$

Then $$\left\{\begin{array}{l} x^2-xy = 6 \\ x^2+y^2 = 61 \end{array}\right\} = \left\{\begin{array}{ll} r^2y^2-ry^2 = 6, & (3) \\ r^2y^2+y^2 = 61. & (4) \end{array}\right\}$$

From (3), $$y^2 = \frac{6}{r^2-r}. \quad (5)$$

From (4), $$y^2 = \frac{61}{r^2+1}. \quad (6)$$

Whence, $$\frac{6}{r^2-r} = \frac{61}{r^2+1}.$$

Simplifying, $$55r^2-61r = 1.$$

Completing square and extracting square root,

$$55r-\tfrac{61}{2} = \pm\tfrac{71}{2}.$$

$$r = 1\tfrac{1}{5} \text{ or } -\tfrac{1}{11}.$$

Substituting in (6), $$y^2 = \frac{61}{(1\frac{1}{5})^2+1} \text{ or } \frac{61}{(-\frac{1}{11})^2+1}.$$

Whence, $$y = \pm 5 \text{ or } \pm \tfrac{11}{2}\sqrt{2}.$$

Hence, $$x = \pm 6 \text{ or } \mp \tfrac{1}{2}\sqrt{2}.$$

13. $$4x^2-2xy = 12; \quad (1) \qquad 2y^2+3xy = 8. \quad (2)$$

Let $$x = ry.$$

Then $$\left.\begin{array}{r} 4x^2-2xy = 12 \\ 2y^2+3xy = 8 \end{array}\right\} = \left\{\begin{array}{r} 4r^2y^2-2ry^2 = 12, \\ 2y^2+3ry^2 = 8. \end{array}\right. \begin{array}{r} (3) \\ (4) \end{array}$$

From (3), $$y^2 = \frac{6}{2r^2-r}. \quad (5)$$

From (4), $$y^2 = \frac{8}{3r+2}. \quad (6)$$

Hence, $$\frac{6}{2r^2-r} = \frac{8}{3r+2}.$$

Simplifying, $$8r^2-13r = 6.$$

Completing square and extracting square root,

$$16r-13 = \pm 19.$$

Whence, $$r = 2 \text{ or } -\tfrac{3}{8}.$$

Substituting in (6), $$y^2 = \frac{8}{3(2)+2} \text{ or } \frac{8}{3(-\frac{3}{8})+2}.$$

Whence, $$y = \pm 1 \text{ or } \pm \tfrac{8}{7}\sqrt{7}.$$

Hence, $$x = \pm 2 \text{ or } \mp \tfrac{3}{7}\sqrt{7}.$$

14. $$x^2+2xy+3y^2 = 17; \quad (1) \qquad 2x^2+3xy+5y^2 = 28. \quad (2)$$

Let $$x = ry.$$

Then $$\left.\begin{array}{r} x^2+2xy+3y^2 = 17 \\ 2x^2+3xy+5y^2 = 28 \end{array}\right\} = \left\{\begin{array}{r} r^2y^2+2ry^2+3y^2 = 17, \\ 2r^2y^2+3ry^2+5y^2 = 28. \end{array}\right. \begin{array}{r} (3) \\ (4) \end{array}$$

From (3), $$y^2 = \frac{17}{r^2+2r+3}. \quad (5)$$

From (4), $$y^2 = \frac{28}{2r^2+3r+5}. \quad (6)$$

Hence, $$\frac{17}{r^2+2r+3}=\frac{28}{2r^2+3r+5}.$$

Simplifying, $$6r^2-5r=-1.$$

Completing square and extracting square root,

$$12r-5=\pm 1.$$

Whence, $$r=\tfrac{1}{2}\text{ or }\tfrac{1}{3}.$$

Substituting in (6), $$y^2=\frac{28}{2(\frac{1}{2})^2+3(\frac{1}{2})+5}\text{ or }\frac{28}{2(\frac{1}{3})^2+3(\frac{1}{3})+5}.$$

Whence, $$y=\pm 2\text{ or }\pm\tfrac{3}{2}\sqrt{2}.$$

Hence, $$x=\pm 1\text{ or }\pm\tfrac{1}{2}\sqrt{2}.$$

15. $$6(x^2-xy)=7xy-6y^2;\quad (1)\qquad x^2-y^2=20.\quad (2)$$

(1) = $$6x^2-13xy+6y^2=0. \tag{3}$$

From (2), $$x^2=20+y^2.$$

Substituting value of x^2 in (3),

$$6(20+y^2)-13xy+6y^2=0,$$

or, $$12y^2-13xy=-120. \tag{4}$$

Let $$x=ry.$$

Then $$\left.\begin{cases} x^2-y^2=20, & (2)\\ 12y^2-13xy=-120, & (4)\end{cases}\right\}=\left\{\begin{array}{ll} r^2y^2-y^2=20, & (5)\\ 12y^2-13ry^2=-120. & (6)\end{array}\right.$$

From (5), $$y^2=\frac{20}{r^2-1}. \tag{7}$$

From (6), $$y^2=\frac{-120}{12-13r}. \tag{8}$$

Hence, $$\frac{20}{r^2-1}=\frac{-120}{12-13r}.$$

Simplifying, $$6r^2-13r=-6.$$

Completing square and extracting square root,

$$12r-13=\pm 5.$$

Whence, $$r=1\tfrac{1}{2}\text{ or }\tfrac{2}{3}.$$

Substituting in (7), $y^2 = \dfrac{20}{(1\frac{1}{2})^2-1}$ or $\dfrac{20}{(\frac{2}{3})^2-1}\cdot$

Whence, $y = \pm 4$ or $\pm 6\sqrt{-1}.$

Hence, $x = \pm 6$ or $\pm 4\sqrt{-1}.$

16. $x^2+xy = 77$; (1) $xy-y^2 = 12.$ (2)

Let $x = ry.$

Then $\left\{\begin{array}{l} x^2+xy = 77 \\ xy-y^2 = 12 \end{array}\right\} = \left\{\begin{array}{r} r^2y^2+ry^2 = 77, \\ ry^2-y^2 = 12. \end{array}\right\}$ (3) (4)

From (3), $y^2 = \dfrac{77}{r^2+r}\cdot$ (5)

From (4), $y^2 = \dfrac{12}{r-1}\cdot$ (6)

Hence, $\dfrac{77}{r^2+r} = \dfrac{12}{r-1}\cdot$

Simplifying, $12r^2-65 = -77.$

Completing square and extracting square root,

$$12r-\tfrac{65}{2} = \pm\tfrac{23}{2}.$$

Whence, $r = 3\frac{2}{3}$ or $1\frac{3}{4}.$

Substituting in (6), $y^2 = \dfrac{12}{3\frac{2}{3}-1}$ or $\dfrac{12}{1\frac{3}{4}-1}\cdot$

Whence, $y = \pm 4$ or $\pm \frac{3}{2}\sqrt{2}.$

Hence, $x = \pm 7$ or $\pm \frac{11}{2}\sqrt{2}.$

17. $5x^2-3xy = 56$; (1) $5y^2+xy = 28.$ (2)

Let $x = ry.$

Then $\left\{\begin{array}{l} 5x^2-3xy = 56 \\ 5y^2+\ \ xy = 28 \end{array}\right\} = \left\{\begin{array}{r} 5r^2y^2-3ry^2 = 56, \\ 5y^2+ry^2 = 28. \end{array}\right\}$ (3) (4)

From (3), $y^2 = \dfrac{56}{5r^2-3r}\cdot$ (5)

From (4), $y^2 = \dfrac{28}{5+r}\cdot$ (6)

Hence, $$\frac{56}{5r^2-3r}=\frac{28}{5+r}.$$

Simplifying, $$140r^2-140r=280,$$

or, $$r^2-r=2.$$

Completing square and extracting square root,

$$r-\tfrac{1}{2}=\pm\tfrac{3}{2}.$$

Hence, $$r=2 \text{ or } -1.$$

Substituting in (6), $$y^2=\frac{28}{5+2} \text{ or } \frac{28}{5+(-1)}.$$

Whence, $$y=\pm 2 \text{ or } \pm\sqrt{7}.$$

Hence, $$x=\pm 4 \text{ or } \mp\sqrt{7}.$$

18. $2x^2-3xy=56$; (1) $\qquad xy-y^2=15.$ (2)

Let $$x=ry.$$

Then $$\left\{\begin{matrix} 2x^2-3xy=56 \\ xy-y^2=15 \end{matrix}\right\}=\left\{\begin{matrix} 2r^2y^2-3ry^2=56, \\ ry^2-y^2=15. \end{matrix}\right\} \begin{matrix} (3) \\ (4) \end{matrix}$$

From (3), $$y^2=\frac{56}{2r^2-3r}. \qquad (5)$$

From (4), $$y^2=\frac{15}{r-1}. \qquad (6)$$

Hence, $$\frac{56}{2r^2-3r}=\frac{15}{r-1}.$$

Simplifying, $$30r^2-101r=-56.$$

Completing square and extracting square root,

$$30r-\tfrac{101}{2}=\pm\tfrac{59}{2}.$$

Whence, $$r=2\tfrac{2}{3} \text{ or } \tfrac{7}{10}.$$

Substituting in (6), $$y^2=\frac{15}{2\frac{2}{3}-1} \text{ or } \frac{15}{\frac{7}{10}-1}.$$

Whence, $$y=\pm 3 \text{ or } \pm 5\sqrt{-2}.$$

Hence, $$x=\pm 8 \text{ or } \pm \tfrac{7}{2}\sqrt{-2}.$$

19. $3xy-4x^2=2; \quad (1) \qquad x^2+y^2=5. \quad (2)$

Let $\qquad x=ry.$

Then $\begin{cases} 3xy-4x^2=2 \\ x^2+y^2=5 \end{cases} = \begin{cases} 3ry^2-4r^2y^2=2, & (3) \\ r^2y^2+y^2=5. & (4) \end{cases}$

From (3), $\qquad y^2=\dfrac{2}{3r-4r^2}. \qquad (5)$

From (4), $\qquad y^2=\dfrac{5}{r^2+1}. \qquad (6)$

Hence, $\qquad \dfrac{2}{3r-4r^2}=\dfrac{5}{r^2+1}.$

Simplifying, $\qquad 22r^2-15r=-2.$

Completing square and extracting square root,

$$22r-\tfrac{15}{2}=\pm\tfrac{7}{2}.$$

Whence, $\qquad r=\frac{1}{2}$ or $\frac{2}{11}.$

Substituting in (6), $\qquad y^2=\dfrac{5}{(\frac{1}{2})^2+1}$ or $\dfrac{5}{(\frac{2}{11})^2+1}.$

Whence, $\qquad y=\pm 2$ or $\pm 2\frac{1}{5}.$

Hence, $\qquad x=\pm 1$ or $\pm\frac{2}{5}.$

20. $3x^2-3xy+y^2=21; \quad (1) \qquad x^2-2xy+3y^2=19. \quad (2)$

Let $\qquad x=ry.$

Then $\begin{cases} 3x^2-3xy+y^2=21 \\ x^2-2xy+3y^2=19 \end{cases} = \begin{cases} 3r^2y^2-3ry^2+y^2=21, & (3) \\ r^2y^2-2ry^2+3y^2=19. & (4) \end{cases}$

From (3), $\qquad y^2=\dfrac{21}{3r^2-3r+1}. \qquad (5)$

From (4), $\qquad y^2=\dfrac{19}{r^2-2r+3}. \qquad (6)$

Hence, $\qquad \dfrac{21}{3r^2-3r+1}=\dfrac{19}{r^2-2r+3}.$

Simplifying, $\qquad 36r^2-15r=44.$

Completing square and extracting square root,

$$36r-\tfrac{15}{2}=\pm\tfrac{81}{2}.$$

Whence, $\qquad r=1\frac{1}{3}$ or $-\frac{11}{12}.$

Substituting in (6), $y^2 = \dfrac{19}{(1\frac{1}{3})^2 - 2(1\frac{1}{3}) + 3}$ or $\dfrac{19}{(-\frac{11}{12})^2 - 2(-\frac{11}{12}) + 3}$.

Whence, $y = \pm 3$ or $\pm \frac{12}{43}\sqrt{43}$.

Hence, $x = \pm 4$ or $\mp \frac{11}{43}\sqrt{43}$.

Art. 344.

1. $x + y = 20$; (1) $\qquad x^2 + y^2 = 202$. (2)

Squaring (1), $x^2 + 2xy + y^2 = 400$.
(2) = $x^2 \qquad + y^2 = 202$.

Subtracting, $2xy = 198$. (3)

Subtracting (3) from (2), $x^2 - 2xy + y^2 = 4$.
Extracting square root, $x - y = \pm 2$. (4)
Adding (1) and (4), $2x = 22$ or 18.
Whence, $x = 11$ or 9.
Hence, $y = 9$ or 11.

2. $x^2 + y^2 = 34$; (1) $\qquad xy = 15$. (2)
Multiplying (2) by 2, $2xy = 30$. (3)
Subtracting (3) from (1), $x^2 - 2xy + y^2 = 4$.
Extracting square root, $x - y = \pm 2$. (4)
Adding (3) and (1), $x^2 + 2xy + y^2 = 64$.
Extracting square root, $x + y = \pm 8$. (5)
Adding (5) and (4), $2x = \pm 10$ or ± 6.
Whence, $x = \pm 5$ or ± 3.
Hence, $y = \pm 3$ or ± 5.

3. $x - y = 2$; (1) $\qquad x^2 + y^2 = 34$. (2)
Squaring (1), $x^2 - 2xy + y^2 = 4$. (3)
Subtracting (3) from (2), $2xy = 30$. (4)
Adding (4) and (2), $x^2 + 2xy + y^2 = 64$.
Extracting square root, $x + y = \pm 8$. (5)
Adding (1) and (5), $2x = 10$ or -6.
Whence, $x = 5$ or -3.
Hence, $y = 3$ or -5.

4. $\sqrt{x}+\sqrt{y}=6$; (1) $\quad x+y=20.$ (2)

Squaring (1), $\quad x+2\sqrt{xy}+y=36.$ (3)

Subtracting (2) from (3), $\quad 2\sqrt{xy}=16.$ (4)

Subtracting (4) from (2), $\quad x-2\sqrt{xy}+y=4.$

Extracting square root, $\quad \sqrt{x}-\sqrt{y}=\pm 2.$ (5)

Adding (1) and (5), $\quad 2\sqrt{x}=8 \text{ or } 4.$

Whence, $\quad \sqrt{x}=4 \text{ or } 2,$

and $\quad x=16 \text{ or } 4.$

Hence, $\quad y=4 \text{ or } 16.$

5. $x+y=xy$; (1) $\quad x^2+y^2=5x^2y^2.$ (2)

Squaring (1), $\quad x^2+2xy+y^2=x^2y^2.$ (3)

Subtracting (2) from (3), $\quad 4x^2y^2=-2xy,$

or, $\quad 2xy=-1.$ (4)

Substituting in (1) and (2), $\quad x+y=-\frac{1}{2},$ (5)

$x^2+y^2=\frac{5}{4}.$ (6)

Subtracting (4) from (6), $\quad x^2-2xy+y^2=\frac{9}{4}.$

Extracting square root, $\quad x-y=\pm\frac{3}{2}.$ (7)

Adding (7) and (5), $\quad 2x=1 \text{ or } -2.$

Whence, $\quad x=\frac{1}{2} \text{ or } -1.$

Hence, $\quad y=-1 \text{ or } \frac{1}{2}.$

6. $x+y=5$; (1) $\quad x^3+y^3=35.$ (2)

Let $x=m+n$ and $y=m-n$.

Then $\quad x+y=(m+n)+(m-n)=2m,$

and $\quad x^3+y^3=(m+n)^3+(m-n)^3=2m^3+6mn^2.$

Hence, $\quad 2m=5,$ and $m=\frac{5}{2}.$

And $\quad 2m^3+6mn^2=35.$ (3)

Substituting $\frac{5}{2}$ for m in (3), $\quad \frac{250}{8}+15n^2=35.$

Whence, $\quad n=\pm\frac{1}{2}.$

But $\quad x=m+n=\frac{5}{2}\pm\frac{1}{2}=3 \text{ or } 2,$

and $\quad y=m-n=\frac{5}{2}-(\pm\frac{1}{2})=2 \text{ or } 3.$

7. $x-y=1$; (1) $x^3-y^3=37$. (2)

Let $x=m+n$ and $y=m-n$.

Then $x-y=(m+n)-(m-n)=2n$,

and $x^3-y^3=(m+n)^3-(m-n)^3=2n^3+6m^2n$.

Hence, $2n=1$, and $n=\frac{1}{2}$.

And $2n^3+6m^2n=37$. (3)

Substituting $\frac{1}{2}$ for n in (3), $\frac{1}{4}+3m^2=37$.

Whence, $m=\pm\frac{7}{2}$.

But $x=m+n=\frac{1}{2}\pm\frac{7}{2}=4$ or -3,

and $y=m-n=\frac{1}{2}-(\pm\frac{7}{2})=3$ or -4.

8. $x^2+y^2=81-2xy$; (1) $x^2+y^2=9+2xy$. (2)

Or, $x^2+2xy+y^2=81$; (1) $x^2-2xy+y^2=9$. (2)

Extracting square roots, $x+y=\pm 9$; (3)

$x-y=\pm 3$. (4)

Adding (3) and (4), $2x=\pm 9\pm 3$.

Whence, $x=\pm 6$ or ± 3.

Hence, $y=\pm 3$ or ± 6.

9. $xy+y^2=2$; (1) $x^2+xy=12$. (2)

Or, $y(x+y)=2$; (3) $x(x+y)=12$. (4)

From (3), $x+y=\dfrac{2}{y}$.

From (4), $x+y=\dfrac{12}{x}$.

Whence, $\dfrac{2}{y}=\dfrac{12}{x}$, or $x=6y$.

Substituting in (1), $6y^2+y^2=2$, or $y^2=\frac{2}{7}$.

Extracting square root, $y=\pm\sqrt{\frac{2}{7}}$ or $\pm\frac{1}{7}\sqrt{14}$.

Hence, $x=\pm\frac{6}{7}\sqrt{14}$.

10. $x^2+y^2-(x+y)=18$; (1) $xy+x+y=19$. (2)

Adding (2) $\times$ 2 to (1), $(x+y)^2+(x+y)=56$.

Completing the square, $(x+y)^2+(x+y)+\frac{1}{4}=\frac{225}{4}$.

Extracting the square root, $x+y+\frac{1}{2}=\pm\frac{15}{2}$.

Whence, $x+y = -8 \text{ or } 7.$ (3)

Substituting values of $(x+y)$ in (1),

$$x^2+y^2-\begin{pmatrix}-8\\7\end{pmatrix} = 18.$$

Whence, $x^2+y^2 = 10 \text{ or } 25.$ (4)

Substituting values of $(x+y)$ in (2),

$$xy+\begin{pmatrix}-8\\7\end{pmatrix} = 19.$$

Whence, $xy = 27 \text{ or } 12,$

and $2xy = 54 \text{ or } 24$ (5)

Subtracting (5) from (4), $x^2-2xy+y^2 = -44 \text{ or } 1.$

Extracting square root, $x-y = \pm 2\sqrt{-11} \text{ or } \pm 1.$ (6)

Adding (6) and (3), $2x = -8\pm 2\sqrt{-11}, 8, \text{ or } 6.$

Whence, $x = -4 \pm \sqrt{-11}, 4, \text{ or } 3.$

Hence, $y = -4 \mp \sqrt{-11}, 3, 4.$

11. $x+\sqrt{xy}+y = 14;$ (1) $x^2+xy+y^2 = 84.$ (2)

Transposing and squaring (1), $x^2+2xy+y^2 = 196-28\sqrt{xy}+xy,$

or, $x^2+xy+y^2 = 196-28\sqrt{xy}.$ (3)

From (2) and (3), $196-28\sqrt{xy} = 84.$

Whence, $\sqrt{xy} = 4,$

and $xy = 16.$

Substituting these values in (1) and (2),

$x+y+4 = 10,$ (4)

and $x^2+y^2+16 = 84.$ (5)

Or, $x+y = 6,$

and $x^2+y^2 = 68.$

Squaring (4), $x^2+2xy+y^2 = 36.$

Subtracting (5) from (4), $2xy = -32.$ (6)

Subtracting (6) from (5), $x^2-2xy+y^2 = 100.$

Extracting square root, $x-y = \pm 10.$ (7)

Adding (7) and (4), $x = 8 \text{ or } 2.$

Whence, $y = 2 \text{ or } 8.$

12. $x^2+y^2=2xy+1$; (1) $x^2+xy+y^2=37$. (2)

Extracting square root of (1),	$x-y=\pm 1$.	(3)
Subtracting (2) from (1),	$-3xy=-36$,	
or,	$xy=12$.	(4)
Adding (4) and (2),	$x^2+2xy+y^2=49$.	
Extracting square root,	$x+y=\pm 7$.	(5)
Adding (5) and (3),	$2x=\pm 8$ or ± 6.	
Whence,	$x=\pm 4$ or ± 3.	
Hence,	$y=\pm 3$ or ± 4.	

13. $x^2-xy+y^2=7$; (1) $x^4+x^2y^2+y^4=133$. (2)

Squaring (1),	$x^4-2x^3y+3x^2y^2-2xy^3+y^4=49$.	(3)
Subtracting (2) from (3),		
	$-2x^3y+2x^2y^2-2xy^3=-84$,	
or,	$xy(x^2-xy+y^2)=42$.	(4)
Multiplying (1) by xy,	$xy(x^2-xy+y^2)=7xy$.	(5)
From (4) and (5),	$7xy=42$.	
Whence,	$xy=6$,	(6)
and	$3xy=18$.	(7)
Subtracting (6) from (1),	$x^2-2xy+y^2=1$.	
Extracting square root,	$x-y=\pm 1$.	(8)
Adding (1) and (7),	$x^2+2xy+y^2=25$.	
Extracting square root,	$x+y=\pm 5$.	(9)
Adding (8) and (9),	$2x=\pm 6$ or ± 4.	
Whence,	$x=\pm 3$ or ± 2.	
Hence,	$y=\pm 2$ or ± 3.	

14. $x^2y+xy^2=20$; (1) $x+y=\frac{5}{4}xy$. (2)

Multiplying (2) by xy,	$x^2y+xy^2=\frac{5}{4}x^2y^2$.	(3)
From (1) and (3),	$\frac{5}{4}x^2y^2=20$.	
Whence,	$xy=\pm 4$.	(4)
Substituting in (1),	$4(x+y)=\pm 20$,	
or,	$x+y=\pm 5$.	(5)
Multiplying (4) by 4,	$4xy=\pm 16$.	(6)

Squaring (5), $x^2+2xy+y^2 = 25.$ (7)

Subtracting (6) from (7), $x^2-2xy+y^2 = 9.$

Extracting square root, $x-y = \pm 3.$ (8)

Adding (5) and (8), $2x = \pm 8$ or $\pm 2.$

Whence, $x = \pm 4$ or $\pm 1.$

Hence, $y = \pm 1$ or $\pm 4.$

15. $x^2+y^2 = 20;$ (1) $(x+y)^2+(x-y)^2 = \frac{5}{2}(x^2-y^2).$ (2)

(2) = $(x^2+2xy+y^2)+(x^2-2xy+y^2) = \frac{5}{2}(x^2-y^2).$

Whence, $4(x^2+y^2) = 5(x^2-y^2).$ (3)

Multiplying (1) by 4, $4(x^2+y^2) = 80.$ (4)

From (3) and (4), $5(x^2-y^2) = 80,$

or, $x^2-y^2 = 16.$ (5)

Adding (5) and (1), $2x^2 = 36,$

or, $x^2 = 18.$

Extracting square root, $x = \pm\sqrt{18}$ or $\pm 3\sqrt{2}.$

Whence, $y = \pm\sqrt{2}.$

16. $x+y = 72;$ (1) $x^{\frac{1}{3}}+y^{\frac{1}{3}} = 6.$ (2)

Cubing (2), $x+3x^{\frac{2}{3}}y^{\frac{1}{3}}+3x^{\frac{1}{3}}y^{\frac{2}{3}}+y = 216.$ (3)

Subtracting (1) from (3), $3x^{\frac{2}{3}}y^{\frac{1}{3}}+3x^{\frac{1}{3}}y^{\frac{2}{3}} = 144,$

or, $x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{1}{3}}+y^{\frac{1}{3}}) = 48.$ (4)

Multiplying (2) by $x^{\frac{1}{3}}y^{\frac{1}{3}}$, $x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{1}{3}}+y^{\frac{1}{3}}) = 6x^{\frac{1}{3}}y^{\frac{1}{3}}$ (5)

From (4) and (5), $6x^{\frac{1}{3}}y^{\frac{1}{3}} = 48,$

or, $\sqrt[3]{xy} = 8.$

Whence, $xy = 512,$

and $4xy = 2048.$ (6)

Squaring (1), $x^2+2xy+y^2 = 5184.$ (7)

Subtracting (6) from (7), $x^2-2xy+y^2 = 3136.$

Extracting square root, $x-y = \pm 56.$ (8)

Adding (8) and (1), $2x = 128$ or $16.$

Whence, $x = 64$ or $8.$

Hence, $y = 8$ or $64.$

17. $x+y-\sqrt{xy} = 7$; (1) $x^2+y^2+xy = 133$. (2)

Transposing and squaring (1), $x^2+2xy+y^2 = 49+14\sqrt{xy}+xy$,

or, $x^2+xy+y^2 = 49+14\sqrt{xy}$. (3)

From (2) and (3), $49+14\sqrt{xy} = 133$,

or, $\sqrt{xy} = 6$,

and $xy = 36$.

Substituting in (1) and (2), $x+y = 13$, (4)

$x^2+y^2 = 97$. (5)

Squaring (4), $x^2+2xy+y^2 = 169$. (6)

Subtracting (5) from (6), $2xy = 72$. (7)

Subtracting (7) from (5), $x^2-2xy+y^2 = 25$.

Extracting square root, $x-y = \pm 5$. (8)

Adding (4) and (8), $2x = 18$ or 8.

Whence, $x = 9$ or 4.

Hence, $y = 4$ or 9.

18. $x^2+y^2 = 13$, (1) $2xy-(x+y) = 7$. (2)

Adding (1) and (2), $(x^2+2xy+y^2)-(x+y) = 20$.

Completing square, $(x+y)^2-(x+y)+\frac{1}{4} = \frac{81}{4}$.

Extracting square root, $x+y-\frac{1}{2} = \pm\frac{9}{2}$.

Whence, $x+y = 5$ or -4. (3)

Substituting in (2), $2xy-\begin{pmatrix}5\\-4\end{pmatrix} = 7$.

Whence, $2xy = 12$ or 3. (4)

Subtracting (4) from (1), $x^2-2xy+y^2 = 10$ or 1.

Extracting square root, $x-y = \pm\sqrt{10}$ or ± 1. (5)

Adding (5) and (3), $2x = \pm\sqrt{10}-4$, 6, or 4.

Whence, $x = \pm\frac{1}{2}\sqrt{10}-2$, 3, or 2.

Hence, $y = \mp\frac{1}{2}\sqrt{10}-2$, 2, or 3.

19. $xy(x+y) = 30$, (1) $\qquad x^3+y^3 = 35.$ (2)

Let $x = m+n$ and $y = m-n$.

Then $\qquad xy(x+y) = (m^2-n^2)2m = 2m^3-2mn^2,$

and $\qquad x^3+y^3 = (m+n)^3+(m-n)^3 = 2m^3+6mn^2.$

Hence, $\qquad 2m^3-2mn^2 = 30,$ (3)

and $\qquad 2m^3+6mn^2 = 35.$ (4)

Multiplying (3) by 3, $\qquad 6m^3-6mn^2 = 90.$ (5)

Adding (4) and (5), $\qquad 8m^3 = 125.$

Extracting cube root, $\qquad 2m = 5,$

and $\qquad m = \frac{5}{2}.$

Substituting this value in (3), $\qquad \frac{125}{4}-5n^2 = 30.$

Whence, $\qquad n = \pm\frac{1}{2}.$

But $\qquad x = m+n = \frac{5}{2}\pm\frac{1}{2} \quad = 3 \text{ or } 2,$

and $\qquad y = m-n = \frac{5}{2}-(\pm\frac{1}{2}) = 2 \text{ or } 3.$

20. $4(x+y) = 3xy$; (1) $\qquad x+y+x^2+y^2 = 26.$ (2)

From (1), $\qquad x+y = \frac{3}{4}xy.$ (3)

Substituting in (2), $\qquad \frac{3}{4}xy+x^2+y^2 = 26.$

Completing square, $\qquad 2xy+x^2+y^2 = 26+\frac{5}{4}xy.$

Extracting square root, $\qquad x+y = \sqrt{26+\frac{5}{4}xy}.$ (4)

From (3) and (4), $\qquad \sqrt{26+\frac{5}{4}xy} = \frac{3}{4}xy.$

Squaring, $\qquad 26+\frac{5}{4}xy = \frac{9}{16}x^2y^2,$

or $\qquad \frac{9}{16}x^2y^2-\frac{5}{4}xy = 26.$

Completing square, $\qquad \frac{9}{16}x^2y^2-\frac{5}{4}xy+\frac{25}{36} = \frac{961}{36}$

Extracting square root, $\qquad \frac{3}{4}xy-\frac{5}{6} = \pm\frac{31}{6}$

Whence, $\qquad xy = 8 \text{ or } -\frac{52}{9}.$ (5)

Substituting in (3), $\qquad x+y = 6 \text{ or } -\frac{13}{3}.$ (6)

Squaring, $\qquad x^2+2xy+y^2 = 36 \text{ or } \frac{169}{9}.$ (7)

Multiplying (5) by 4, $\qquad 4xy = 32 \text{ or } -\frac{208}{9}.$ (8)

Subtracting (7) from (6), $\qquad x^2-2xy+y^2 = 4 \text{ or } \frac{377}{9}.$

Extracting square root, $\qquad x-y = \pm 2 \text{ or } \pm\frac{1}{3}\sqrt{377}.$ (9)

Adding (9) and (6), $\qquad 2x = 8,\ 4, \text{ or } -\frac{13}{3}\pm\frac{1}{3}\sqrt{377}.$

Whence, $\qquad x = 4,\ 2, \text{ or } \frac{1}{6}(-13\pm\sqrt{377}).$

Hence, $\qquad y = 2,\ 4, \text{ or } \frac{1}{6}(-13\mp\sqrt{377}).$

Art. 345.

1. $4(x^2-y^2) = 15;$ (1) $\qquad 2(x-y) = 3.$ (2)

Or, $x^2-y^2 = \frac{15}{4};$ $\qquad x-y = \frac{3}{2}.$

Dividing (1) by (2), $x+y = \frac{5}{2}.$ (3)

From (2) and (3), $2x = 4,$ and $2y = 1.$

Whence, $x = 2,$ and $y = \frac{1}{2}.$

2. $x^2-y^2 = 102;$ (1) $\qquad x-y = 3.$ (2)

Dividing (1) by (2), $x+y = 34.$ (3)

From (2) and (3), $2x = 37,$ and $2y = 31.$

Whence, $x = 18\frac{1}{2},$ and $y = 15\frac{1}{2}.$

3. $4x^2-9y^2 = 76;$ (1) $\qquad 2x-3y = 2.$ (2)

Dividing (1) by (2), $2x+3y = 38.$ (3)

From (2) and (3), $4x = 40,$ and $6y = 36.$

Whence, $x = 10,$ and $y = 6.$

4. $x+y = 74;$ (1) $\qquad x^2+y^2 = 2740.$ (2)

Squaring (1), $x^2+2xy+y^2 = 5476.$ (3)

Subtracting (2) from (3), $2xy = 2736.$ (4)

Subtracting (4) from (2), $x^2-2xy+y^2 = 4.$

Extracting square root, $x-y = \pm 2.$ (5)

Adding (5) and (1), $x = 38$ or $36.$

Hence, $y = 36$ or $38.$

5. $x-y = 4;$ (1) $\qquad x^2+y^2 = 1066.$ (2)

Squaring (1), $x^2-2xy+y^2 = 16.$ (3)

Subtracting (3) from (2), $2xy = 1050.$ (4)

Adding (4) and (2), $x^2+2xy+y^2 = 2116.$

Extracting square root, $x+y = \pm 46.$ (5)

Adding (1) and (5), $x = 25$ or $-21.$

Hence, $y = 21$ or $-25.$

6. $x+y = 7xy$; (1) $\quad 12xy = 1.$ (2)

From (2), $\quad xy = \frac{1}{12}.$

Substituting in (1), $\quad x+y = \frac{7}{12}.$ (3)

Squaring (3), $\quad x^2+2xy+y^2 = \frac{49}{144}.$ (4)

From (2), $\quad 4xy = \frac{1}{3}.$ (5)

Subtracting (5) from (4), $\quad x^2-2xy+y^2 = \frac{1}{144}.$

Extracting square root, $\quad x-y = \pm\frac{1}{12}.$ (6)

Adding (3) and (6), $\quad x = \frac{1}{3}$ or $\frac{1}{4}.$

Hence, $\quad y = \frac{1}{4}$ or $\frac{1}{3}.$

7. $x+y = 12$; (1) $\quad xy = 20.$ (2)

Squaring (1), $\quad x^2+2xy+y^2 = 144.$ (3)

From (2), $\quad 4xy = 80.$ (4)

Subtracting (4) from (3), $\quad x^2-2xy+y^2 = 64.$

Extracting square root, $\quad x-y = \pm 8.$ (5)

Adding (1) and (5), $\quad x = 10$ or $2.$

Hence, $\quad y = 2$ or $10.$

8. $x-y = 1$; (1) $\quad xy = 240.$ (2)

Squaring (1), $\quad x^2-2xy+y^2 = 1.$ (3)

From (2), $\quad 4xy = 960.$ (4)

Adding (3) and (4), $\quad x^2+2xy+y^2 = 961.$

Extracting square root, $\quad x+y = \pm 31.$ (5)

Adding (1) and (5), $\quad x = 16$ or $-15.$

Hence, $\quad y = 15$ or $-16.$

9. $x^2+xy+2y^2 = 44$; (1) $\quad 2x^2-xy+y^2 = 16.$ (2)

Let $\quad x = ry.$

Then
$$\left.\begin{array}{r} x^2+xy+2y^2 = 44 \\ 2x^2-xy+y^2 = 16 \end{array}\right\} = \left\{\begin{array}{lr} r^2y^2+ry^2+2y^2 = 44, & (3) \\ 2r^2y^2-ry^2+y^2 = 16. & (4) \end{array}\right.$$

From (3), $$y^2 = \frac{44}{r^2+r+2}.$$

From (4), $$y^2 = \frac{16}{2r^2-r+1}.$$

Hence, $$\frac{44}{r^2+r+2} = \frac{16}{2r^2-r+1}.$$

Clearing of fractions, transposing, and uniting,

$$72r^2-60r = -12,$$

or, $$36r^2-30r = -6.$$

Completing square and extracting square root,

$$r = \tfrac{1}{2} \text{ or } \tfrac{1}{3}.$$

Substituting in (4), $$y^2 \doteq \frac{16}{2\left(\frac{1}{4}\right)-\frac{1}{2}+1} \text{ or } \frac{16}{2\left(\frac{1}{3}\right)^2-\frac{1}{3}+1}.$$

Whence, $$y = \pm 4 \text{ or } \pm 3\sqrt{2}.$$

Hence, $$x = \pm 2 \text{ or } \pm \sqrt{2}.$$

10. $$2x^2+3xy+y^2 = 70; \quad (1) \qquad 6x^2+xy-y^2 = 50. \quad (2)$$

Let $$x = ry.$$

Then $$\left.\begin{cases} 2x^2+3xy+y^2 = 70 \\ 6x^2+\ xy-y^2 = 50 \end{cases}\right\} = \left\{\begin{matrix} 2r^2y^2+3ry^2+y^2 = 70, & (3) \\ 6r^2y^2+\ ry^2-y^2 = 50. & (4) \end{matrix}\right.$$

From (3), $$y^2 = \frac{70}{2r^2+3r+1}. \quad (5)$$

From (4), $$y^2 = \frac{50}{6r^2+r-1}. \quad (6)$$

Hence, $$\frac{70}{2r^2+3r+1} = \frac{50}{6r^2+r-1}.$$

Clearing of fractions, transposing, and uniting,

$$320r^2-80r = 120,$$

or, $$16r^2-4r = 6.$$

Completing square and extracting square root,

$$8r-1 = \pm 5.$$

Whence, $$r = \tfrac{3}{4} \text{ or } -\tfrac{1}{2}.$$

Substituting in (5), $$y^2 = \frac{70}{2\left(\frac{3}{4}\right)^2+3\left(\frac{3}{4}\right)+1} \text{ or } \frac{70}{2\left(-\frac{1}{2}\right)^2+3\left(-\frac{1}{2}\right)+1}.$$

Whence, $$y = \pm 4 \text{ or } \infty.$$

Hence, $$x = \pm 3 \text{ or } \infty.$$

11. $x^2+xy+y^2 = 37$; (1) $x^4+x^2y^2+y^4 = 481$. (2)

Dividing (2) by (1), $x^2-xy+y^2 = 13$. (3)

Subtracting (3) from (1), $2xy = 24$,

$xy = 12$. (4)

Adding (4) to (1), $x^2+2xy+y^2 = 49$.

Extracting square root, $x+y = \pm 7$. (5)

Subtracting (4)×3 from (1), $x^2-2xy+y^2 = 1$.

Extracting square root, $x-y = \pm 1$. (6)

Adding (5) and (6), $2x = \pm 8$ or ± 6.

Hence, $x = \pm 4$ or ± 3.

Whence, $y = \pm 3$ or ± 4.

12. $x^2+y^2-1 = 2xy$; (1) $xy(xy+1) = 6$. (2)

Or, $x^2-2xy+y^2 = 1$; (1) $x^2y^2+xy = 6$. (2)

Let $x = m+n$, and $y = m-n$.

Then, substituting in (1), $4n^2 = 1$.

Whence, $n = \pm \frac{1}{2}$.

Substituting values of x and y in (2),

$$m^4+\tfrac{1}{2}m^2-\tfrac{3}{16} = 6.$$

Whence, $16m^4+8m^2 = 99$.

Completing square and extracting square root,

$$4m^2+1 = \pm 10.$$

Whence, $m^2 = -\frac{11}{4}$ or $\frac{9}{4}$.

Extracting square root, $m = \pm \frac{1}{2}\sqrt{-11}$ or $\pm \frac{3}{2}$.

But $x = m+n = \pm \frac{1}{2}+(\pm \frac{1}{2}\sqrt{-11})$ or $\pm \frac{1}{2} \pm \frac{3}{2}$

$= \pm \frac{1}{2}(\pm \sqrt{-11} \pm 1)$ or ± 2.

And $y = m-n = (\pm \frac{1}{2}\sqrt{-11}) \mp \frac{1}{2}$ or $\pm \frac{3}{2} \mp \frac{1}{2}$

$= \frac{1}{2}(\pm \sqrt{-11} \mp 1)$ or ± 1.

13. $x^2+3xy=54;\quad(1)\qquad xy+4y^2=115.\quad(2)$

Let $$x=ry.$$

Then $$\left.\begin{array}{l}x^2+3xy=\ \ 54\\ xy+4y^2=115\end{array}\right\}=\left\{\begin{array}{ll}r^2y^2+3ry^2=\ \ 54, & (3)\\ ry^2+\ 4y^2=115. & (4)\end{array}\right.$$

From (3), $$y^2=\frac{54}{r^2+3r}.\quad(5)$$

From (4), $$y^2=\frac{115}{r+4}.\quad(6)$$

Hence, $$\frac{54}{r^2+3r}=\frac{115}{r+4}.$$

Clearing of fractions, transposing, and uniting,

$$115r^2+291r=216.$$

Completing square and extracting square root,

$$230r+291=\pm 429.$$

Whence, $$r=-\tfrac{72}{23}\text{ or }\tfrac{69}{115}.$$

Substituting in (6), $$y^2=\frac{115}{-\frac{72}{23}+4}\text{ or }\frac{115}{\frac{69}{115}+4}.$$

Whence, $$y=\pm 11\tfrac{1}{2}\text{ or }\pm 5.$$

Hence, $$x=\mp 36\text{ or }\pm 3.$$

14. $x^2+xy-8x=3;\quad(1)\qquad y^2+xy-8y=6.\quad(2)$

Or, $x(x+y-8)=3;\quad(1)\qquad y(x+y-8)=6.\quad(2)$

From (1), $$x+y-8=\frac{3}{x}.$$

From (2), $$x+y-8=\frac{6}{y}.$$

Hence, $$\frac{3}{x}=\frac{6}{y},\quad\text{or}\quad y=2x.$$

Substituting in (2), $$6x^2-16x=6,$$

or, $$3x^2-8x=3.$$

Completing square and extracting square root,

$$6x-8=\pm 10.$$

Whence, $$x=3\text{ or }-\tfrac{1}{3}.$$

Hence, $$y=6\text{ or }-\tfrac{2}{3}.$$

15. $x-y=1;$ (1) $x^2-xy+y^2=21.$ (2)

Squaring (1), $x^2-2xy+y^2=1.$ (3)

Subtracting (3) from (2), $xy=20.$

Whence, $3xy=60.$ (4)

Adding (4) and (2), $x^2+2xy+y^2=81.$

Extracting square root, $x+y=\pm 9.$ (5)

Adding (5) and (1), $2x=10$ or $8.$

Whence, $x=5$ or $4.$

Hence, $y=-4$ or $-5.$

16. $5(x^2-y^2)=4(x^2+y^2);$ (1) $x+y=8.$ (2)

Uniting and transposing in (1), $x^2-9y^2=0.$

Whence, $x=\pm 3y.$

Substituting in (2), $\pm 3y+y=8.$

Whence, $y=2$ or $-4.$

Hence, $x=6$ or $12.$

17. $4x+9y=30;$ (1) $2x^2-xy=3y^2.$ (2)

Squaring (1), $16x^2+72xy+81y^2=900.$ (3)

Multiplying (2) by 8, $16x^2-8xy-24y^2=0.$ (4)

Subtracting (4) from (3), $80xy+105y^2=900,$

or, $16xy+21y^2=180.$ (5)

Multiplying (1) by $4y$, $16xy+36y^2=120y.$ (6)

Subtracting (5) from (6), $15y^2-120y=-180,$

or, $y^2-8y=-12.$

Completing square and extracting square root,

$$y-4=\pm 2.$$

Whence, $y=6$ or $2.$

Hence, $x=-6$ or $3.$

18. $(x-6)^2+(y-5)^2+2xy = 101; \quad (1) \qquad 2x = 3y-2. \quad (2)$

From (2), $$x = \frac{3y-2}{2}.$$

Substituting in (1),

$$\left(\frac{3y-2}{2}-6\right)^2+(y-5)^2+2y\left(\frac{3y-2}{2}\right) = 101.$$

Simplifying, $$25y^2-132y = 108.$$

Completing square and extracting square root,

$$5y-\tfrac{66}{5} = \pm\tfrac{84}{5}.$$

Whence, $$y = 6 \text{ or } -\tfrac{18}{25}.$$

Hence, $$x = 8 \text{ or } -\tfrac{52}{25}.$$

19. $4x^2+2xy+\dfrac{y^2}{4}+\dfrac{5(4x+y)}{12} = 41; \quad (1) \qquad 4x = y+4. \quad (2)$

From (2), $$x = \frac{y+4}{4}.$$

Simplifying (1), $$\left(\frac{4x+y}{2}\right)^2+\frac{20x+5y}{12} = 41.$$

Substituting value of x and simplifying,

$$(y+2)^2+\frac{5y+10}{6} = 41.$$

Simplifying, $$6y^2+29y = 212.$$

Completing square and extracting square root,

$$12y+29 = \pm 77.$$

Whence, $$y = 4 \text{ or } -8\tfrac{5}{6}.$$

Hence, $$x = 2 \text{ or } -\tfrac{29}{24}.$$

20. $3x+2y = 5xy; \quad (1) \qquad 15x-4y = 4. \quad (2)$

From (2), $$x = \frac{4+4y}{15}.$$

Substituting in (1), $$3\left(\frac{4+4y}{15}\right)+2y = 5y\left(\frac{4+4y}{15}\right).$$

Simplifying, $$10y^2-11y = 6.$$

Completing square and extracting square root,

$$20y-11 = \pm 19.$$

Whence, $$y = \tfrac{3}{2} \text{ or } -\tfrac{2}{5}.$$

Hence, $$x = \tfrac{2}{3} \text{ or } \tfrac{4}{25}.$$

Art. 346.

1. Let x = first number, and y = second number.

Then will	$x+y = 20,$	(1)
and	$xy = 99.$	(2)
Multiplying (2) by 4,	$4xy = 396.$	(3)
Squaring (1),	$x^2+2xy+y^2 = 400.$	(4)
Subtracting (3) from (4),	$x^2-2xy+y^2 = 4.$	
Extracting square root,	$x-y = \pm 2.$	(5)
Adding (1) and (5),	$2x = 22.$	
Whence,	$x = 11$ or $9.$	
Hence,	$y = 9$ or $11.$	

2. Let x = No. rods in length, and y = No. rods in width.

Then	$2(x+y) = 104,$	(1)
and	$xy = 3(160).$	(2)
Squaring (1),	$x^2+2xy+y^2 = 2704.$	(3)
Multiplying (2) by 4,	$4xy = 1920.$	(4)
Subtracting (4) from (3),	$x^2-2xy+y^2 = 784.$	
Extracting square root,	$x-y = \pm 28.$	(5)
Adding (1) and (5),	$2x = 80.$	
Whence,	$x = 40$ or $12.$	
Hence,	$y = 12$ or $40.$	

3. Let x = No. rods in length, and y = No. rods in width.

Then will	$2(x+y) = 112,$	(1)
and	$xy = 720.$	(2)
Multiplying (2) by 4,	$4xy = 2880.$	(3)
Squaring (1),	$x^2+2xy+y^2 = 3136.$	(4)
Subtracting (3) from (4),	$x^2-2xy+y^2 = 256.$	
Extracting square root,	$x-y = \pm 16.$	(5)
Adding (1) and (5),	$2x = 72.$	
Whence,	$x = 36$ or $20,$	
and	$y = 20$ or $36.$	

4. Let x = first number, and y = second number.

Then will	$x-y = 11,$	(1)
and	$xy = 276.$	(2)
Multiplying (2) by 4,	$4xy = 1104.$	(3)
Squaring (1),	$x^2-2xy+y^2 = 121.$	(4)
Adding (3) and (4),	$x^2+2xy+y^2 = 1225.$	
Extracting square root,	$x+y = \pm 35.$	(5)
Adding (1) and (5),	$2x = 46.$	
Whence,	$x = 23.$	
Hence,	$y = 12.$	

5. Let x = No. rods in length, and y = No. rods in width.

Then will	$x-y = 12,$	(1)
and	$xy = 18 \times 160.$	(2)
Multiplying (2) by 4,	$4xy = 11520.$	(3)
Squaring (1),	$x^2-2xy+y^2 = 144.$	(4)
Adding (3) and (4),	$x^2+2xy+y^2 = 11664.$	
Extracting square root,	$x+y = \pm 108.$	(5)
Adding (1) and (5),	$2x = 120.$	
Whence,	$x = 60.$	
Hence,	$y = 48.$	

6. Let x = first number, and y = second number.

Then will	$x+y = 14,$	(1)
and	$x^3+y^3 = 854.$	(2)
Dividing (2) by (1),	$x^2-xy+y^2 = 61.$	(3)
Squaring (1),	$x^2+2xy+y^2 = 196.$	(4)
Subtracting (3) from (4),	$3xy = 135.$	
Whence,	$xy = 45.$	(5)
Subtracting (5) from (3),	$x^2-2xy+y^2 = 16.$	
Extracting square root,	$x-y = \pm 4.$	(6)
Adding (1) and (6),	$2x = 18 \text{ or } 10.$	
Whence,	$x = 9 \text{ or } 5.$	
Hence,	$y = 5 \text{ or } 9.$	

7. Let $\quad x =$ length of first block,

and $\quad y =$ length of second block.

Then will $$x+y = 12, \tag{1}$$

and $$x^3+y^3 = 468. \tag{2}$$

Dividing (2) by (1), $$x^2-xy+y^2 = 39. \tag{3}$$

Squaring (1), $$x^2+2xy+y^2 = 144. \tag{4}$$

Subtracting (3) from (4), $$3xy = 105.$$

Whence, $$xy = 35. \tag{5}$$

Subtracting (5) from (3), $$x^2-2xy+y^2 = 4.$$

Extracting square root, $$x-y = \pm 2. \tag{6}$$

Adding (1) and (6), $$2x = 14 \text{ or } 10.$$

Whence, $$x = 7 \text{ or } 5.$$

Hence, $$y = 5 \text{ or } 7.$$

8. Let $x =$ A's rate per hour, and $y =$ B's rate per hour.

Then will $$\frac{36}{x} = \frac{36}{y} - 3, \tag{1}$$

and $$\frac{43\frac{1}{3}}{x+\frac{1}{3}} = \frac{43\frac{1}{3}}{y+\frac{1}{3}} - 3. \tag{2}$$

Simplifying (1) and (2), $$12x-12y = xy, \tag{3}$$

and $$127x-133y+9xy = 1. \tag{4}$$

Multiplying (3) by 9, $$108x-108y+9xy = 0. \tag{5}$$

Subtracting (5) from (4), $$19x-25y = 1.$$

Whence, $$y = \frac{19x-1}{25}.$$

Substituting in (3), $$12x-12\left(\frac{19x-1}{25}\right) = x\left(\frac{19x-1}{25}\right).$$

Simplifying, $$19x^2-73x = -12.$$

Completing square and extracting square root,
$$38x-73 = \pm 79.$$

Whence, $$x = 4.$$

Hence, $$y = 3.$$

9. Let x = number of feet in circumference of fore-wheel,

and y = number of feet in circumference of hind-wheel.

Then will $$\frac{270}{x} - 2 = \frac{270}{y}, \tag{1}$$

and $$\frac{396}{x+3} - 2 = \frac{396}{y+3}. \tag{2}$$

Simplifying (1) and (2), $$135y - 135x = xy, \tag{3}$$

and $$195y - 201x - xy = 9. \tag{4}$$

Subtracting (3) from (4), $$20y - 22x = 3.$$

Whence, $$x = \frac{20y-3}{22}.$$

Substituting in (3), $$135y - 135\left(\frac{20y-3}{22}\right) = y\left(\frac{20y-3}{22}\right).$$

Simplifying, $$20y^2 - 273y = 405.$$

Completing square and extracting square root,

$$40y - 273 = \pm 327$$

Whence, $$y = 15.$$

Hence, $$x = 13\tfrac{1}{2}.$$

10. Let x = No. miles between A and B,

and y = No. days before they met.

Then will $\frac{x+30}{2y}$ = No. miles that A could travel in a day,

and $\frac{x-30}{2y}$ = No. miles that B could travel in a day.

Then will $$9\left(\frac{x-30}{2y}\right) = \frac{x+30}{2}, \tag{1}$$

and $$4\left(\frac{x+30}{2y}\right) = \frac{x-30}{2}. \tag{2}$$

Simplifying (1) and (2), $$4x - xy + 30y = -120, \tag{3}$$

and $$9x - xy - 30y = 270. \tag{4}$$

Adding (3) and (4), $$13x - 2xy = 150. \tag{5}$$

Subtracting (3) from (4), $$x - 12y = 78. \tag{6}$$

Whence, $$x = 78 + 12y.$$

Substituting in (5),

$$13(78+12y)-2y(78+12y) = 150.$$

Simplifying, $y^2 = 36,$

and $y = \pm 6.$

Substituting in (6), $x-72 = 78.$

Whence, $x = 150.$

11. Let $x =$ No. rods in length, and $y =$ No. rods in width.

Then $xy = 1575,$ (1)

and $(x-5)(y-5) = 1200.$ (2)

(2) = $xy-5x-5y = 1175,$

or, $5x-xy+5y = -1175.$ (3)

Adding (1) and (3), $5x+5y = 400.$

Whence, $x+y = 80.$ (4)

Squaring (4), $x^2+2xy+y^2 = 6400.$ (5)

Multiplying (1) by 4, $4xy = 6300.$ (6)

Subtracting (6) from (5), $x^2-2xy+y^2 = 100.$

Extracting square root, $x-y = \pm 10.$ (7)

Adding (4) and (7), $2x = 90$ or $70.$

Whence, $x = 45$ or $35.$

Hence, $y = 35$ or $45.$

12. Let $x =$ digit in units' place,

$y =$ digits in tens' place,

and $10y+x =$ the number.

Then will $\dfrac{10y+x}{xy} = 2,$ (1)

and $10y+x+27 = 10x+y.$ (2)

From (1), $10y-2xy+x = 0.$ (3)

From (2), $x-y = 3.$ (4)

Subtracting (4) from (3), $11y-2xy = -3.$

Whence, $y = \dfrac{3}{2x-11}.$

Substituting in (4), $x-\dfrac{3}{2x-11} = 3.$

Whence, $2x^2-17x = 30.$

Completing square and extracting square root,

$$4x-17=7.$$

Whence, $x=6.$

Hence, $y=3.$

13. Let $x =$ number of acres,

and $\frac{420}{x} =$ number of dollars raised as rent per acre.

Then will $$\left(\frac{420}{x}+2\tfrac{1}{2}\right)(x-4)=420.$$

Simplifying, $5x^2-20x=3360,$

or, $x^2-4x=672.$

Completing square and extracting square root,

$$x-2=26.$$

Whence, $x=28.$

Hence, $\frac{420}{x}=15.$

14. Let $x =$ number of yards in circumference of fore-wheel,

and $y =$ number of yards in circumference of hind-wheel.

Then will $$\frac{120}{x}-6=\frac{120}{y}, \qquad (1)$$

and $$\frac{120}{x+1}-4=\frac{120}{y+1}. \qquad (2)$$

Simplifying (1), $20y-xy-20x=0.$ (3)

Simplifying (2), $29y-31x-xy=1.$ (4)

Subtracting (3) from (4), $9y-11x=1.$

Whence, $y=\frac{1+11x}{9}.$

Substituting in (3),

$$20\left(\frac{1+11x}{9}\right)-x\left(\frac{1+11x}{9}\right)-20x=0.$$

Simplifying, $11x^2-39x=20.$

Completing square and extracting square root,

$$22x-39=49.$$

Whence, $x=4.$

Hence, $y=5.$

15. Let $x =$ price in cents per pound of mace,

and $y =$ price in cents per pound of cloves.

Then will $$80x + 100y = 65, \quad (1)$$

and $$\frac{20}{y} - 60 = \frac{10}{x}. \quad (2)$$

Simplifying (1), $$16x + 20y = 13. \quad (3)$$

Simplifying (2), $$2x - 6xy - y = 0. \quad (4)$$

From (3), $$x = \frac{13 - 20y}{16}.$$

Substituting in (4),

$$2\left(\frac{13-20y}{16}\right) - 6y\left(\frac{13-20y}{16}\right) - y = 0.$$

Simplifying, $$120y^2 - 134 = -26,$$

or, $$10y^2 - 67 = -13.$$

Completing square and extracting square root,

$$120y - 67 = \mp 37.$$

Whence, $$y = .25.$$

Hence, $$x = .50.$$

16. Let $x^2 =$ number of square rods in A's field,

$2.25x^2 =$ value of A's field in dollars,

$y^2 =$ number of square rods in B's field,

and $2.25y^2 =$ value of B's field in dollars.

Then will $$4x + 4y = 200,$$

or, $$x + y = 50, \quad (1)$$

and $$x^2 + y^2 = 1300. \quad (2)$$

Squaring (1), $$x^2 + 2xy + y^2 = 2500. \quad (3)$$

Subtracting (2) from (3), $$2xy = 1200. \quad (4)$$

Subtracting (4) from (2), $$x^2 - 2xy + y^2 = 100.$$

Extracting square root, $$x - y = \pm 10. \quad (5)$$

Adding (1) and (5), $$x = 30 \text{ or } 20.$$

Whence, $$2.25x^2 = 2025 \text{ or } 900.$$

Hence, $$2.25y^2 = 900 \text{ or } 2025.$$

17. Let $x =$ the first number, and $y =$ the second number.

Then will $(x-y)(x^2-y^2) = 32,$ (1)

and $(x+y)(x^2+y^2) = 272.$ (2)

Simplifying (1) and (2), $x^3-x^2y-xy^2+y^3 = 32,$ (3)

and $x^3+x^2y+xy^2+y^3 = 272.$ (4)

Adding (3) and (4), $2x^3+2y^3 = 304,$

or, $x^3+y^3 = 152.$ (5)

Subtracting (3) from (4), $2x^2y+2xy^2 = 240,$

or, $xy(x+y) = 120.$ (6)

Multiplying (6) by 3, $3x^2y+3xy^2 = 360.$ (7)

Adding (5) and (7), $x^3+3x^2y+3xy^2+y^3 = 512.$

Extracting cube root, $x+y = 8.$ (8)

Substituting in (6), $8xy = 120.$

Whence, $4xy = 60.$ (9)

Squaring (8), $x^2+2xy+y^2 = 64.$ (10)

Subtracting (9) from (10), $x^2-2xy+y^2 = 4.$

Extracting square root, $-y = \pm 2.$ (11)

Adding (8) and (11), $x = 5$ or $3.$

Whence, $y = 3$ or $5.$

18. Let $x =$ the first number, and $y =$ the second number.

Then will $x-y = 2,$ (1)

and $\left(\frac{x}{y}\right)^2 + 4\left(\frac{x}{y}\right) = 9\frac{4}{9}.$ (2)

Completing square and extracting square root of (2),

$$\frac{x}{y} + 2 = \frac{11}{3}.$$

Clearing of fractions, transposing, and uniting,

$$x = \tfrac{5}{3}y.$$

Substituting in (1), $\frac{5}{3}y-y = 2.$

Whence, $y = 3,$

Hence, $x = 5.$

19. Let $x =$ the less number, and $y =$ the greater number.

Then will $x(x+y) = 36,$ (1)

and $y(x+y) = 45.$ (2)

Adding (1) and (2), $x^2+2xy+y^2 = 81.$

Extracting square root, $x+y = \pm 9.$ (3)

Substituting in (1), $\pm 9x = 36.$

Whence, $x = \pm 4.$

Hence, $y = \pm 5.$

20. Let $x =$ the first number, and $y =$ the second number.

Then will $xy+(x+y) = 47,$ (1)

and $(x^2+y^2)-(x+y) = 62.$ (2)

Multiplying (1) by 2, $2xy+2(x+y) = 94.$ (3)

Adding (2) and (3), $(x+y)^2+(x+y) = 156.$

Completing square and extracting square root,

$$x+y+\tfrac{1}{2} = \tfrac{25}{2}.$$

Whence, $x+y = 12.$ (4)

Substituting in (1), $xy+12 = 47.$

Whence, $4xy = 140.$ (5)

Squaring (4), $x^2+2xy+y^2 = 144.$ (6)

Subtracting (5) from (6), $x^2-2xy+y^2 = 4.$

Extracting square root, $x-y = 2.$ (7)

Adding (4) and (7), $x = 7.$

Hence, $y = 5.$

21. Let $x =$ acres in first, and $140-x =$ acres in second.

Let $y =$ amount each received.

Then will $x\left(\dfrac{y}{140-x}\right) = 1800,$ (1)

and $(140-x)\left(\dfrac{y}{x}\right) = 3200.$ (2)

Simplifying (1) and (2), $xy+1800x = 252000,$ (3)

and $140y-xy-3200x = 0.$ (4)

Adding (3) and (4), $140y-1400x = 252000,$

or, $y-10x = 1800.$

Whence, $y = 1800 + 10x$.

Substituting in (3), $x(1800 + 10x) + 1800x = 252000$.

Whence, $x^2 + 360x = 25200$.

Completing square and extracting square root,

$$x + 180 = 240.$$

Whence, $x = 60$.

Hence, $y = 80$.

22. Let $x =$ number of sheep,

and $\frac{136}{x} =$ number of dollars paid for each sheep.

Then will $(x-22)\left(\frac{136}{x} + 1\right) = 138$.

Simplifying, $x^2 - 24x = 2992$.

Completing square and extracting square root,

$$x - 12 = 56.$$

Whence, $x = 68$.

Hence, $\frac{136}{x} = 2$.

23. Let $x =$ number of dollars in smaller sum,

$12000 - x =$ number of dollars in larger sum,

and $y =$ rate per cent.

Then will $\frac{12000 - x}{x} = 100y$, (1)

and $y(12000 - x) - 210 = x$. (2)

Simplifying (1) and (2), $100xy + x = 12000$, (3)

and $-x - xy + 12000y = 210$. (4)

From (3), $x = \frac{12000}{100y + 1}$.

Substituting in (4),

$$-\frac{12000}{100y+1} - y\left(\frac{12000}{100y+1}\right) + 12000y = 210.$$

Simplifying, $40000y^2 - 700y = 407$. (5)

Completing square and extracting square root,

$$80000y-700 = 8100.$$

Whence, $y = .11.$

Hence, $x = 1000,$

and $12000-1000 = 11000.$

24. Let $x =$ number of dollars paid for each ox,

and $y =$ number of dollars paid for each sheep.

Then will $$60x+80y = 5300, \quad (1)$$

and $$\frac{450}{y} - 42 = \frac{225}{x}. \quad (2)$$

Simplifying (1) and (2), $$3x+4y = 265, \quad (3)$$

and $$150x-14xy-75y = 0. \quad (4)$$

From (3), $$x = \frac{265-4y}{3}.$$

Substituting in (4),

$$150\left(\frac{265-4y}{3}\right) - 14y\left(\frac{265-4y}{3}\right) - 75y = 0.$$

Simplifying, $$56y^2-4535y = -39750.$$

Completing square and extracting square root,

$$112y-4535 = \pm 3415.$$

Whence, $y = 10.$

Hence, $x = 75.$

25. Let $x =$ first price;

then $2250-x =$ second price.

Also, $$\frac{100 \times 62\frac{1}{2}}{2250-x} = \text{first rate};$$

and $$\frac{100 \times 40}{x} = \text{second rate}.$$

So that $$\frac{62\frac{1}{2}x}{100(2250-x)} = \frac{40(2250-x)}{100x}.$$

Whence, $$125x^2 = 80(2250-x)^2,$$

or, $$\frac{25x^2}{16} = (2250-x)^2.$$

Extracting square root, $$\frac{5x}{4} = 2250-x.$$

Whence, $$x = 1000.$$

Hence, $$\frac{100 \times 62\frac{1}{2}}{2250-1000} = 5;$$

and $$\frac{4000}{1000} = 4.$$

Art. 379.

5. Let x = first part, and $60-x$ = second part.

Then $$x(60-x) : x^2+(60-x)^2 :: 2 : 5.$$

$$5x(60-x) = 2[x^2+(60-x)^2].$$

Whence, $$x^2-60x = -800.$$

Completing the square, $$x^2-60x+900 = 100.$$

Extracting the square root, $$x-30 = \pm 10.$$

Hence, $$x = 40 \text{ or } 20,$$

and $$60-x = 20 \text{ or } 40.$$

6. $$6x+a : 3x-b :: 4x+b : 2x-a.$$

$$(6x+a)(2x-a) = (3x-b)(4x+b).$$

$$12x^2-4ax-a^2 = 12x^2-bx-b^2.$$

$$(4a-b)x = b^2-a^2.$$

$$x = \frac{b^2-a^2}{4a-b}.$$

7. $x^3-y^3 : (x-y)^3 :: 61 : 1;$ (1) $x : 8 :: 40 : y.$ (2)

From (1), $\dfrac{3x^2y-3xy^2}{(x-y)^3} = 60.$

From (2), $xy = 320,$

and $x = \dfrac{320}{y}.$

Whence, $\dfrac{xy}{(x-y)^2} = 20.$

Substituting value of xy, $\dfrac{320}{(x-y)^2} = 20.$

Dividing by 20, $\dfrac{16}{(x-y)^2} = 1.$

Whence, $x-y = \pm 4.$

Substituting value of x, $\dfrac{320}{y} - y = \pm 4.$

Whence, $y^2 \pm 4y = 320.$

Hence, $y \pm 2 = \pm 18,$

and $y = \pm 16 \text{ or } \pm 20.$

$x = \pm 20 \text{ or } \pm 16.$

8. $\frac{2}{7} : \frac{3}{4} :: \frac{5}{6} : x.$

$\frac{2}{7}x = \frac{3}{4} \times \frac{5}{6},$

$\frac{2}{7}x = \frac{5}{8}.$

Whence, $16x = 35,$ and $x = 2\frac{3}{16}.$

9. Let $x =$ the number.

Then $9 : 20 :: x : 100.$

Whence, $20x = 900,$ and $x = 45.$

10. $x : 10 :: 33 : 16.$ Whence, $x = 20\frac{5}{8}.$

$22 : x :: 33 : 16.$ Whence, $x = 10\frac{2}{3}.$

$22 : 10 :: x : 16.$ Whence, $x = 35\frac{1}{5}.$

$22 : 10 :: 33 : x.$ Whence, $x = 15.$

11. $x+y : x :: 7 : 5.$

Whence, $5x+5y = 7x,$

and $y = \dfrac{2x}{5}.$

Substituting value of x in $xy+y^2 = 126,$

$$\frac{2x^2}{5}+\frac{4x^2}{25} = 126.$$

Whence, $14x^2 = 3150,$

$$x^2 = 225, \quad x = \pm 15, \quad \text{and} \quad y = \pm 6.$$

12. Let x = greater number, and y = less number.

Then will $x : y :: x+y : 42,$ and $x : y :: x-y : 6.$

From first proportion, $\dfrac{x+y}{y} = \dfrac{x+y+42}{42}.$

Whence, $42x+42y = xy+y^2+42y,$

$$(42-y)x = y^2,$$

and $x = \dfrac{y^2}{42-y}.$

From second proportion, $\dfrac{x+y}{y} = \dfrac{x-y+6}{6}.$

Whence, $6x+6y = xy-y^2+6y,$

$$(6-y)x = -y^2,$$

and $x = \dfrac{-y^2}{6-y}.$

Then $\dfrac{y^2}{42-y} = \dfrac{-y^2}{6-y},$

or, $\dfrac{1}{42-y} = \dfrac{-1}{6-y}.$

Whence, $6-y = -42+y.$

Hence, $y = 24,$ and $x = 32.$

13. Let x^2 = area of smallest square,

and $(x+10)^2$ = area of largest square.

Then will $x^2 : (x+10)^2 :: 9 : 25.$

Extracting square root, $x : x+10 :: 3 : 5.$

Whence, $x = 15,$ side of smallest square,

$x+10 = 25,$ side of largest square.

14. Let $x = \text{one number.}$

Then will $\frac{135}{x} = \text{the other number.}$

Hence, $x^2 - \frac{18225}{x^2} : x^2 - 270 + \frac{18225}{x^2} :: 4 : 1.$

Whence, $x^2 - \frac{18225}{x^2} = 4x^2 - 1080 + \frac{12900}{x^2},$

$$3x^4 - 1080x^2 = -91125,$$

and $x^4 - 360x^2 = -30375.$

Whence, $x^2 = 225.$

Hence, $x = 15,$

and $\frac{135}{x} = 9.$

15. Let $x =$ No. dollars A had, and $y =$ No. dollars B had.

Then will $x+150 : y-50 :: 3 : 2,$ (1)

and $x-50 : y+100 :: 5 : 9.$ (2)

From (1), $2x-3y = -450.$ (3)

From (2), $x = \frac{5y+950}{9}.$ (4)

Substituting this value of x in (3),

$$\frac{10y+1900}{9} - 3y = -450.$$

Whence, $y = 350.$

And $x = 300.$

16. $$\frac{\sqrt{m+x}+\sqrt{m-x}}{\sqrt{m+x}-\sqrt{m-x}} = \frac{n}{1}.$$

By (7), Art. 300, and Art. 373,

$$\sqrt{m+x} : \sqrt{m-x} :: n+1 : n-1.$$

Squaring, $m+x : m-x :: (n+1)^2 : (n-1)^2.$

Whence, $x(n-1)^2 + x(n+1)^2 = m(n+1)^2 - m(n-1)^2.$

Reducing and factoring, $2(n^2+1)x = 4mn,$

and $x = \frac{2mn}{n^2+1}.$

17. $$\frac{\sqrt[3]{x+1}+\sqrt[3]{x-1}}{\sqrt[3]{x+1}-\sqrt[3]{x-1}} = 2.$$

Whence, $$\sqrt[3]{x+1}+\sqrt[3]{x-1} = 2\sqrt[3]{x+1}-2\sqrt[3]{x-1}.$$

$$3\sqrt[3]{x-1} = \sqrt[3]{x+1}.$$

Cubing, $$27x-27 = x+1.$$

Whence, $$26x = 28, \quad \text{and} \quad x = 1\tfrac{1}{13}.$$

18. Let x = first digit, y = second digit, and z = third digit.

Then will $$x : y :: y : z. \qquad (1)$$

$$100x+10y+z : x+y+z :: 124 : 7. \qquad (2)$$

$$100x+10y+z+594 = 100z+10y+x.$$

From this, $$x-z = -6. \qquad (3)$$

From (1), $$y^2 = 6x+x^2,$$

and $$y = \sqrt{6x+x^2}. \qquad (4)$$

From (2), $$\frac{100x+10y+z}{x+y+z} = \frac{124}{7}.$$

By (4), Art. 300, and dividing by 9, $$\frac{11x+y}{x+y+z} = \frac{13}{7};$$

and by (3), Art. 300, $$\frac{10x-z}{11x+y} = \frac{6}{13}.$$

Whence, $$64x-13z-6y = 0. \qquad (5)$$

$$\begin{array}{lrrl} (3)\times 13 = & 13x-13z & & = -78. \\ \hline \text{Subtracting,} & 51x & -6y & = 78, \\ \text{or,} & 17x & -2y & = 26. \end{array}$$

Substituting value of y as found in (4),

$$17x-2\sqrt{6x+x^2} = 26.$$

Transposing, $$2\sqrt{6x+x^2} = 17x-26.$$

Squaring, transposing, and uniting,

$$285x^2-908x = -676.$$

Completing square and extracting square root,

$$285x-454 = 116.$$

Whence, $x = 2.$

Hence, $y = 4,$ and $z = 8.$

Art. 388.

1. Given $a = 2,\ d = 3,$ and $n = 11$; to find l and S.

$$l = a+(n-1)\,d.$$

Hence, $l = 2+(11-1)\,3$ or $32.$

$$S = \frac{n}{2}(a+l).$$

Hence, $S = \dfrac{11}{2}(2+32)$ or $187.$

2. Given $a = 8,\ l = 203,$ and $d = 5$; to find n and S.

$$l = a+(n-1)\,d.$$

Hence, $203 = 8+(n-1)\,5,$

or, $n = 40.$

$$S = \frac{n}{2}(a+l).$$

Hence, $S = \dfrac{40}{2}(8+203)$ or $4220.$

3. Given $l = 1,\ S = 1717,$ and $n = 34$; to find a and d.

$$S = \frac{n}{2}(a+l).$$

Hence, $1717 = \dfrac{34}{2}(a+1),$

or, $a = 100.$

$$l = a+(n-1)\,d.$$

Hence, $1 = 100+(34-1)\,d,$

or, $d = -3.$

4. Given $a = 1$, $l = 1000$, and $d = 1$; to find S.

$$l = a+(n-1)d.$$

Hence, $$1000 = 1+(n-1)1,$$

or, $$n = 1000.$$

$$S = \frac{n}{2}(a+l).$$

Hence, $$S = \frac{1000}{2}(1+1000) \text{ or } 500500.$$

5. Given $a = 1$, $n = 23$, and $S = 184$; to find d and l.

$$S = \frac{n}{2}(a+l).$$

Hence, $$184 = \frac{23}{2}(1+l),$$

or, $$l = 15.$$

$$l = a+(n-1)d.$$

Hence, $$15 = 1+(23-1)d,$$

or, $$d = \tfrac{7}{11}.$$

6. Given $a = 3$, $l = 38$, and $m = 6$; to find terms.

$$d = \frac{l-a}{m+1}.$$

Hence, $$d = \frac{38-3}{6+1} \text{ or } 5.$$

Hence the series is 3, 8, 13, 18 23, 28, 33, 38.

7. Given $a = \frac{1}{3}$, $l = \frac{1}{2}$, and $m = 3$; to find series.

$$d = \frac{l-a}{m+1}.$$

Hence, $$d = \frac{\frac{1}{2}-\frac{1}{3}}{3+1} \text{ or } \tfrac{1}{24}.$$

Hence the series is $\frac{1}{3}$, $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$, $\frac{1}{2}$.

8. Given $a = -7$, $d = -7$, and $n = 101$; to find S.

$$l = a+(n-1)d.$$

Hence, $\quad l = -7+(101-1)(-7)$ or $-707.$

$$S = \frac{n}{2}(a+l).$$

Hence, $\quad S = \frac{101}{2}[-7+(-707)]$ or $-36057.$

9. Given $a = 100$, $n = 21$, and $S = 1260$; to find d.

$$S = \frac{n}{2}(a+l).$$

Hence, $\quad 1260 = \frac{21}{2}(100+l),$

or, $\quad l = 20.$

$$l = a+(n-1)d.$$

Hence, $\quad 20 = 100+(21-1)d,$

or, $\quad d = -4.$

10. $40 \times 40 =$ number of miles A will travel.

Given $a = 20$, $d = \frac{3}{4}$, and $n = 40$; to find S.

$$l = a+(n-1)d.$$

Hence, $\quad l = 20+(40-1)\frac{3}{4}$ or $\frac{197}{4}.$

$$S = \frac{n}{2}(a+l).$$

Hence, $\quad S = 20(20+\frac{197}{4})$ or $1385.$

$1385 =$ number of miles B will travel.

Hence, $\quad 1600-1385 = 215$, No. miles that A is in advance of B.

11. Given $a = a$, $d = 2a$, and $n = t$; to find l.

$$l = a+(n-1)d.$$

Hence, $\quad l = a+(t-1)2a = (2t-1)a.$

12. Given $a = a$, $d = 2a$, $n = t$, and $l = (2t-1)a$; to find S.

$$S = \frac{n}{2}(a+l).$$

Hence, $\quad S = \frac{t}{2}[a+(2t-1)a]$ or $at^2.$

13. Given $a = 1$, $d = 2$, and $n = 52$; to find l and S.

$$l = a+(n-1)d.$$

Hence.
$$l = 1+(52-1)2 \text{ or } 103.$$

$$S = \frac{n}{2}(a+l).$$

Hence,
$$S = \frac{52}{2}(1+103) \text{ or } 2704.$$

14. Let $x =$ No. years.

Then
$$180x = \frac{x}{2}(270+l). \quad (1)$$

And
$$l = 270+(x-1)60. \quad (2)$$

Substituting in (1) value of l in (2),

$$180x = \frac{x}{2}[270+270+(x-1)60].$$

Whence,
$$30x^2 = 120x; \text{ or, } x = 4.$$

15. Given $a = 2\frac{3}{4}+1\frac{1}{4} = 4$, $d = 1\frac{1}{2}$, and $n = 10$; to find l.

$$l = a+(n-1)d.$$

Hence,
$$l = 4+(10-1)1\frac{1}{2} \text{ or } 17\frac{1}{2}.$$

16. Given $a = 4$, $d = 4$, $n = 100$, and $l = 400$; to find S.

$$S = \frac{n}{2}(a+l).$$

Hence,
$$S = \frac{100}{2}(4+400) \text{ or } 20200 \text{ yds.}$$

$$20200 \text{ yds.} = 11 \text{ mi. } 3 \text{ fur. } 32 \text{ rds. } 4 \text{ yds.}$$

17. Let $x-3y$, $x-y$, $x+y$, and $x+3y$ represent the numbers.

Then will $(x-3y)^2+(x+3y)^2+(x-y)^2+(x+y)^2 = 84.$ (1)

$$(x-3y)(x+3y)(x+y)(x-y) = 105. \quad (2)$$

Simplifying,
$$x^4-10x^2y^2+9y^4 = 105, \quad (3)$$

and
$$x^2+5y^2 = 21. \quad (4)$$

Squaring (4),
$$x^4+10x^2y^2+25y^4 = 441. \quad (5)$$

Subtracting (5) from (3),
$$5x^2y^2+4y^4 = 84. \quad (6)$$

Adding (5) and (3),
$$x^4+17y^4 = 273. \quad (7)$$

Let $$x = ry.$$

Then will $$\left\{\begin{array}{r} 5x^2y^2 + 4y^4 = 84 \\ x^4+17y^4 = 273 \end{array}\right\} = \left\{\begin{array}{r} 5r^2y^4 + 4y^4 = 84, \\ r^4y^4+17y^4 = 273. \end{array}\right\} \quad \begin{array}{r} (8) \\ (9) \end{array}$$

From (8), $$y^4 = \frac{84}{5r^2+4}. \quad (10)$$

From (9), $$y^4 = \frac{273}{r^4+17}. \quad (11)$$

Hence, $$\frac{84}{5r^2+4} = \frac{273}{r^4+17}.$$

Simplifying, $$28r^4-455r^2 = -112.$$

Completing square and extracting square root,

$$56r^2-455 = 441.$$

Whence, $$r^2 = 16,$$

and $$r = 4.$$

Substituting in (10), $$y^4 = \frac{84}{5\,(16)+4} = 1,$$

and $$y = 1.$$

Whence, $$x = 4.$$

Hence, $x-3y = 1$, $x-y = 3$, $x+y = 5$, and $x+3y = 7$.

18. Let $x-2y$, $x-y$, x, $x+y$, and $x+2y$ represent the numbers.

Then will $$(x-2y)+(x-y)+x+(x+y)+(x+2y) = 35, \quad (1)$$

and $$(x-2y)^2+(x-y)^2+x^2+(x+y)^2+(x+2y)^2 = 285. \quad (2)$$

Simplifying, $$5x = 35,$$

or, $$x = 7, \quad (3)$$

and $$x^2+2y^2 = 57. \quad (4)$$

Substituting value of x in (4), $$49+2y^2 = 57.$$

Whence, $$y^2 = 4, \quad \text{and} \quad y = 2.$$

Hence, $x-2y = 3$, $x-y = 5$, $x = 7$, $x+y = 9$, and $x+2y = 11$.

19. Let $x-y$, x, and $x+y$ represent the numbers.

Then will $$(x-y)^2+x^2+(x+y)^2 = 1232, \tag{1}$$

and $$x^2-16 = (x-y)(x+y). \tag{2}$$

Simplifying, $$3x^2+2y^2 = 1232, \tag{3}$$

and $$y^2 = 16, \quad \text{or} \quad y = 4.$$

Substituting value of y in (3), $$3x^2+32 = 1232.$$

Whence, $$x^2 = 400,$$

and $$x = 20.$$

Hence, $x-y = 16$, $x = 20$, and $x+y = 24$.

20. Let $x-3y$, $x-y$, $x+y$, and $x+3y$ represent the numbers.

Then will $$(x-3y)^2+(x+3y)^2 = 4500, \tag{1}$$

and $$(x-y)^2+(x+y)^2 = 4100. \tag{2}$$

Simplifying, $$x^2+9y^2 = 2250, \tag{3}$$

and $$x^2+y^2 = 2050. \tag{4}$$

Subtracting (4) from (3), $$8y^2 = 200,$$

$$y^2 = 25, \quad \text{and} \quad y = 5.$$

Substituting value of y in (4), $$x^2+25 = 2050.$$

Whence, $$x = 45.$$

Hence, $x-3y = 30$, $x-y = 40$, $x+y = 50$, and $x+3y = 60$.

21. Let $x-2y$, $x-y$, x, $x+y$, and $x+2y$ represent the numbers.

Then will $$(x-2y)(x-y)(x)(x+y)(x+2y) = 945, \tag{1}$$

and $$(x-2y)+(x-y)+x+(x+y)+(x+2y) = 25. \tag{2}$$

Simplifying, $$x^5-5x^3y^2+4xy^4 = 945, \tag{3}$$

$$5x = 25,$$

and $$x = 5. \tag{4}$$

Substituting value of x in (3),

$$3125-625y^2+20y^4 = 945.$$

Whence, $$4y^4-125y^2 = -436.$$

Completing square and extracting square root,

$$8y^2-125 = \pm 93.$$

Whence, $$y^2 = 4, \quad \text{and} \quad y = 2.$$

Hence, $x-2y = 1$, $x-y = 3$, $x = 5$, $x+y = 7$, and $x+2y = 9$.

22. Let
$$x-y = \text{hundreds' digit},$$
$$x = \text{tens' digit},$$
$$x+y = \text{units' digit},$$
and
$$100(x-y)+10x+(x+y) = \text{the number}.$$
Then will
$$\frac{100(x-y)+10x+(x+y)}{(x-y)+x+(x+y)} = 26, \quad (1)$$
and
$$100(x-y)+10x+(x+y)+26 = 100(x+y)+10x+(x-y). \quad (2)$$
Simplifying,
$$111x-99y = 78x,$$
or,
$$x = 3y, \quad (3)$$
and
$$111x-99y+198 = 111x+99y,$$
or,
$$y = 1. \quad (4)$$
Substituting in (3),
$$x = 3.$$
Hence, $x-y = 2$, $x = 3$, and $x+y = 4$.

23. Let x, $x+500$, $x+1000$, $x+1500$, $x+2000$, and $x+2500$ represent the number of dollars in the respective shares.

Then will
$$x+(x+500)+(x+1000)+(x+1500)+(x+2000)+(x+2500) = 9000.$$
Whence,
$$6x = 1500,$$
and
$$x = 250.$$
Hence, $x = 250$, $x+500 = 750$, $x+1000 = 1250$, $x+1500 = 1750$, $x+2000 = 2250$, and $x+2500 = 2750$.

24. Given, $a = 193$ inches, $d = 386$, and $n = 60$; to find S.
$$l = a+(n-1)d.$$
Whence,
$$l = 193+(60-1)386 = 22967.$$
$$S = \frac{n}{2}(a+l).$$
Whence,
$$S = \frac{60}{2}(193+22967)$$
$$= 694800 \text{ inches or } 57900 \text{ ft.}$$

Art. 397.

1. Given $a = 3$, $r = 5$, and $n = 5$; to find l.

$$l = ar^{n-1}.$$

Whence, $$l = 3 \times 5^4 \text{ or } 1875.$$

2. Given $a = 6561$, $n = 10$, and $r = \frac{1}{3}$; to find l.

$$l = ar^{n-1}.$$

Whence, $$l = 6561 \times (\tfrac{1}{3})^9 = \tfrac{6561}{19683} \text{ or } \tfrac{1}{3}.$$

3. Given $a = 2$, $r = 3$, and $n = 6$; to find l.

$$l = ar^{n-1}.$$

Whence, $$l = 2 \times 3^5 \text{ or } 486.$$

4. Given $a = 1$, $r = \frac{1}{2}$, and $n = 8$; to find l.

$$l = ar^{n-1}.$$

Whence, $$l = 1 \times (\tfrac{1}{2})^7 = \tfrac{1}{128}.$$

5. Given $a = 3$, $r = 2$, and $n = 13$; to find l.

$$l = ar^{n-1}.$$

Whence, $$l = 3 \times 2^{12} = 12288.$$

6. Given $a = 164$, $r = \frac{1}{2}$, and $n = 13$; to find l.

$$l = ar^{n-1}.$$

Whence, $$l = 164 \times (\tfrac{1}{2})^{12} = \tfrac{164}{4096} \text{ or } \tfrac{41}{1024}.$$

7. Given $a = 3$, $r = 2$, and $n = 8$; to find S.

$$S = \frac{a(r^n - 1)}{r - 1}.$$

Whence, $$S = \frac{3(2^8 - 1)}{2 - 1} \text{ or } 765.$$

8. Given $a = 4$, $r = \frac{1}{3}$, and $n = 7$; to find S.

$$S = \frac{a(r^n - 1)}{r - 1}.$$

Whence, $$S = \frac{4[(\frac{1}{3})^7 - 1]}{\frac{1}{3} - 1} = \tfrac{4372}{729} \text{ or } 5\tfrac{727}{729}.$$

9. Given $a = 5$, $l = 3645$, and $m = 5$; to find r.

$$l = ar^{m+1}.$$

Whence, $$3645 = 5 \times r^6,$$

or, $$r^6 = 729.$$

Extracting sixth root, $$r = 3.$$

Hence, the series is 5, 15, 45, 135, 405, 1215, and 3645.

10. Given $a = 1$ and $r = \frac{1}{2}$; to find S.

$$S = \frac{a}{1-r}.$$

Whence, $$S = \frac{1}{1-\frac{1}{2}} = 2.$$

11. Given $a = \frac{1}{2}$ and $r = \frac{3}{4}$; to find S.

$$S = \frac{a}{1-r}.$$

Whence, $$S = \frac{\frac{1}{2}}{1-\frac{3}{4}} = 2.$$

12. Given $a = 14$ and $r = \frac{1}{7}$; to find S.

$$S = \frac{a}{1-r}.$$

Whence, $$S = \frac{14}{1-\frac{1}{7}} = 16\tfrac{1}{3}.$$

13. Let x, xy, xy^2, and xy^3 represent the four terms.

Then will $$x + xy^2 = 148, \qquad (1)$$

and $$xy + xy^3 = 888. \qquad (2)$$

From (1), $$y^2 + 1 = \frac{148}{x}. \qquad (3)$$

From (2), $$y^2 + 1 = \frac{888}{xy}. \qquad (4)$$

Hence, $$\frac{148}{x} = \frac{888}{xy},$$

or, $$y = 6.$$

Substituting in (1), $$x + 36x = 148.$$

Whence, $$37x = 148, \quad \text{or,} \quad x = 4.$$

Hence the series is 4, 24, 144, and 864.

14. Let x, $\sqrt{xy}$, and y represent the numbers.

Then will $\quad x+\sqrt{xy}+y = 14,$ (1)

and $\quad x^2+xy+y^2 = 84.$ (2)

Dividing (2) by (1), $\quad x-\sqrt{xy}+y = 6.$ (3)

Adding (3) and (1), $\quad x+y = 10.$ (4)

Subtracting (3) from (1), $\quad \sqrt{xy} = 4,$

and $\quad xy = 16.$ (5)

Squaring (4), $\quad x^2+2xy+y^2 = 100.$ (6)

Subtracting 4 times (5) from (6), $x^2-2xy+y^2 = 36.$

Extracting square root, $\quad x-y = 6.$ (7)

Adding (4) and (7), $\quad x = 8.$

Whence, $\quad y = 2.$

Hence the series is 8, 4, and 2.

15. Let x, $\sqrt{xy}$, and y represent the numbers.

Then will $\quad x+\sqrt{xy}+y = 52,$ (1)

and $\quad x+y : \sqrt{xy} :: 10 : 3.$ (2)

From (2), $\quad 3x-10\sqrt{xy}+3y = 0.$ (3)

Multiplying (1) by 3, $\quad 3x+3\sqrt{xy}+3y = 156.$ (4)

Subtracting (3) from (4), $\quad \sqrt{xy} = 12,$

or, $\quad xy = 144.$ (5)

Substituting value of $\sqrt{xy}$ in (1), $\quad x+y = 40.$ (6)

Squaring (6), $\quad x^2+2xy+y^2 = 1600.$ (7)

Subtracting 4 times (5) from (7), $x^2-2xy+y^2 = 1024.$

Extracting square root, $\quad x-y = 32.$ (8)

Adding (6) and (8), $\quad x = 36.$

Whence, $\quad y = 4.$

Hence the series is 4, 12, and 36.

16. Given $a = 50+\frac{1}{2}(50)$ and $r = 1\frac{1}{2}$; to find S.

$$S = \frac{a}{r-1}.$$

Whence, $$S = \frac{75}{\frac{3}{2}-1} = 150.$$

17. Let x, $\sqrt{xy}$, and y represent number of dollars in the respective shares.

Then will $$x+\sqrt{xy}+y=210. \quad (1)$$

$$\sqrt{xy}-30=y. \quad (2)$$

Adding (2) and (1), $$x+2\sqrt{xy}=240. \quad (3)$$

Subtracting (2) from (1), $$x+2y=180. \quad (4)$$

From (4), $$x=180-2y.$$

Substituting in (3),

$$(180-2y)+2\sqrt{y(180-2y)}=240.$$

Simplifying, $$y^2+40y=-300.$$

Completing square and extracting square root,

$$y-20=10.$$

Whence, $$y=30.$$

Substituting in (4), $$x=120.$$

Hence the series is 120, 60, and 30.

Art. 406.

17. $$\frac{x^2+x-12}{x^3-5x^2+7x-3}=\frac{(x+4)\ (x-3)}{(x-1)^2\,(x-3)}=\frac{x+4}{(x-1)^2}.$$

18. $$\frac{x^3+(1+x)\,xy+y^2}{x^4-y^2}=\frac{(x+y)\,(x^2+y)}{(x^2-y)\,(x^2+y)}=\frac{x+y}{x^2-y}.$$

20. $$\frac{(ab-1)^2+(a+b-2)\,(a+b-2ab)}{(ab+1)^2-(a+b)^2}$$

$$=\frac{a^2+4ab-2a+b^2-2b+1-2ab^2-2a^23+a^2b^2}{a^2b^2-a^2-b^2+1}$$

$$=\frac{(ab-a-b+1)\,(ab-a-b+1)}{(ab+a+b+1)\,(ab-a-b+1)}=\frac{ab-a-b+1}{ab+a+b+1}.$$

Art. 407.

19. $$\frac{x^2-yz}{(x+y)(x+z)}+\frac{y^2-xz}{(x+y)(y+z)}+\frac{z^2-xy}{(x+z)(y+z)}$$

$$=\frac{[(x^2-yz)(y+z)]+[(y^2-xz)(x+z)]+[(z^2-xy)(x+y)]}{(x+y)(x+z)(y+z)}=0.$$

20. $$\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(x-z)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}$$

$$=\frac{(x-y+z)(x+y-z)}{(x+z+y)(x+z-y)}+\frac{(y-x+z)(y+x-z)}{(x+y+z)(x+y-z)}+\frac{(z-x+y)(z+x-y)}{(y+z-x)(y+z+x)}$$

$$=\frac{x+y-z}{x+y+z}+\frac{y-x+z}{x+y+z}+\frac{z+x-y}{y+z+x}=\frac{x+y+z}{x+y+z}=1.$$

21. $$\frac{x^2-9x+20}{x^2-6x}\times\frac{x^2-13x+42}{x^2-5x}=\frac{(x-5)(x-4)(x-6)(x-7)}{[x(x-6)][x(x-5)]}$$

$$=\frac{(x-4)(x-7)}{x^2}.$$

22. $$\frac{x+1}{x-1}\times\frac{x^2+x-2}{x^2+x}=\frac{(x+1)(x-1)(x+2)}{(x-1)[x(x+1)]}=\frac{x+2}{x}.$$

23. $$\frac{3x^2-x}{5}\times\frac{10}{2x^2-4x}=\frac{[x(3x-1)]10}{5[2x(x-2)]}=\frac{3x-1}{x-2}.$$

24. $$\frac{a^2-x^2}{a+b}\times\frac{a^2-b^2}{ax+x^2}\times\left(a+\frac{ax}{a-x}\right)=\frac{(a-x)(a+x)(a-b)(a+b)a^2}{(a+b)[x(a+x)](a-x)}$$

$$=\frac{a^3-a^2b}{x}.$$

25. $$\left(1-\frac{a-b}{a+b}\right)\times\left(2+\frac{2b}{a-b}\right)=\frac{[(a+b)-(a-b)][2(a-b)+2b]}{(a+b)(a-b)}$$

$$=\frac{4ab}{a^2-b^2}.$$

26. $$\frac{x(a-x)}{a^2+2ax+x^2}\times\frac{x(a+x)}{a^2-2ax+x^2}=\frac{[x(a-x)][x(a+x)]}{(a+x)(a+x)(a-x)(a-x)}=\frac{x^2}{a^2-x^2}.$$

27. $$\frac{m^2-mn+n^2}{m^3-3mn(m-n)-n^3} \times \frac{m^2-n^2}{m^3+n^3} = \frac{(m^2-mn+n^2)(m-n)(m+n)}{(m-n)^3(m^2-mn+n^2)(m+n)}$$

$$= \frac{1}{(m-n)^2}.$$

28. $$\left\{\frac{2x}{x^2+1}+\frac{2x}{x^2-1}\right\} \div \left\{\frac{x}{x^2+1}-\frac{x}{x^2-1}\right\} = \frac{4x^3}{x^4-1} \times \frac{x^4-1}{-2x} = -2x^2.$$

29. $$\left\{a+b+\frac{b^2}{a}\right\} \div \left\{a+b+\frac{a^2}{b}\right\} = \frac{a^2+ab+b^2}{a} \times \frac{b}{ab+b^2+a^2} = \frac{b}{a}.$$

30. $$\left\{a+\frac{b-a}{1+ab}\right\} \div \left\{1-\frac{a(b-a)}{1+ab}\right\} = \frac{a^2b+b}{1+ab} \times \frac{1+ab}{1+a^2} = b.$$

31. $$\frac{a^3+b^3}{a^2-b^2} \div \frac{a^2-ab+b^2}{a-b} = \frac{(a+b)(a^2-ab+b^2)}{(a+b)(a-b)} \times \frac{a-b}{a^2-ab+b^2} = 1.$$

32. $$\left\{\frac{2a+b}{a+b}-1\right\} \div \left\{1-\frac{b}{a+b}\right\} = \frac{a}{a+b} \times \frac{a+b}{a} = 1.$$

33. $$\frac{\frac{1}{1+x}}{1-\frac{1}{1+x}} + \frac{\frac{1}{1+x}}{\frac{x}{1-x}} + \frac{\frac{1}{1-x}}{\frac{x}{1+x}} = \frac{1}{x} + \frac{1-x}{x(1+x)} + \frac{1+x}{x(1-x)}$$

$$= \frac{(1-x^2)+(1-x)^2+(1+x)^2}{x(1-x^2)} = \frac{3+x^2}{x(1-x^2)}$$

34. $$\frac{b}{b-rx^2} - \left\{\frac{3b}{rx^2-b}+\frac{2brx^2}{r^2x^4-b^2}\right\} = \frac{4b^2+6brx^2}{b^2-r^2x^4}.$$

35. $$\frac{a^2}{(x-a)(a-b)} + \frac{b^2}{(x-b)(b-a)} = \frac{a^2(x-b)+b^2[(x-a)-1]}{(x-a)(x-b)(a-b)}$$

$$= \frac{ax+bx-ab}{(x-a)(x-b)}.$$

36. $$\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}$$

$$= \frac{(b-c)+[-1(a-c)]+(a-b)}{(a-b)(a-c)(b-c)} = 0.$$

37. $$\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-x)(y-z)}-\frac{1}{xyz}$$

$$=\frac{yz(y-z)+xz[-1(x-z)]-(x-y)(x-z)(y-z)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{xy^2-x^2y}{xyz(x-y)(x-z)(y-z)}=\frac{-xy(x-y)}{xyz(x-y)(x-z)(y-z)}$$

$$=\frac{1}{z(x-z)(z-y)}.$$

38. $$\frac{yz}{(x-z)(y-x)}+\frac{xz}{(x-y)(y-z)}+\frac{xy}{(z-y)(x-z)}$$

$$=\frac{yz[-1(y-z)]+xz(x-z)+xy[-1(x-y)]}{(x-z)(x-y)(y-z)}$$

$$=\frac{yz^2-y^2z+x^2z-xz^2+xy^2-x^2y}{x^2y-xy^2+y^2z-x^2z+xz^2-yz^2}=-1.$$

39. $$\frac{x+y}{(x-5)(5-y)}+\frac{x+5}{(x-y)(y-5)}-\frac{y+5}{(x-y)(5-x)}$$

$$=\frac{(x+y)(x-y)+(x+5)(5-x)-(y+5)(y-5)}{(x-5)(5-y)(x-y)}=0.$$

40. $$\frac{(a-b)^2}{a-b-1}+\frac{(a-b)^2}{b-a-1}+\frac{1}{b+1-a}+\frac{1}{a-b+1}$$

$$=\frac{(a-b)^2(-a+b-1)+(a-b)^2(a-b-1)+[-1(-a+b-1)]+[-1(a-b-1)]}{(a-b-1)(-a+b-1)}$$

$$=\frac{-2a^2+4ab-2b^2+2}{-a^2+2ab-b^2+1}=2.$$

41. $$\frac{x-a}{(a+y)(y-x)}+\frac{x+y}{(a+x)(a+y)}+\frac{a-y}{(a+x)(y-x)}$$

$$=\frac{(x-a)(a+x)+(x+y)(y-x)+(a-y)(a+y)}{(a+y)(y-x)(a+x)}=0.$$

42. $$\frac{x^2-6}{x^2-5x+6}-\frac{2x+9}{5(x-3)}-\frac{3x+4}{5(x-2)}$$

$$=\frac{5(x^2-6)-(2x+9)(x-2)-(3x+4)(x-3)}{5(x-3)(x-2)}=0.$$

43. $$\frac{b}{(1-ab)(a-b)}+\frac{a}{(b^2-1)(a-b)}+\frac{ab^3}{(b^2-1)(1-ab)}$$

$$=\frac{b(b^2-1)+a(1-ab)+ab^3(a-b)}{(1-ab)(a-b)(b^2-1)}$$

$$=\frac{b^3-b+a-a^2b+a^2b^3-ab^4}{-a^2b^3-b^3+ab^4-a+a^2b+b}=-1.$$

Art. 409.

3. $$\tfrac{1}{4}\left[3x-\tfrac{2}{3}(1+x)\right]-\frac{1-\frac{1}{5}x}{5\frac{1}{2}}=\frac{2\frac{2}{5}+\frac{1}{25}(x-1)}{2\frac{1}{5}},$$

or, $$\frac{7x-2}{12}+\frac{10-2x}{55}=\frac{x+59}{55}.$$

Clearing of fractions,

$$385x-110+120-24x=12x+708.$$

Whence, $$349x=698,$$

$$x=2.$$

4. $$\frac{6x-7\frac{1}{3}}{13-2x}+2x+\frac{16x+1}{24}=4\tfrac{5}{12}-\frac{12\frac{5}{8}-8x}{3},$$

or, $$\frac{6x-7\frac{1}{3}}{13-2x}+\frac{64x+1}{24}=\frac{53}{12}-\frac{12\frac{5}{8}-8x}{3}.$$

Multiplying by 24,

$$\frac{144x-176}{13-2x}+64x+1=106-101+64x.$$

Whence, $$\frac{144x-176}{13-2x}=4,$$

$$144x-176=42-8x,$$

and $$x=1\tfrac{1}{3}.$$

8. $$\frac{m^2-1}{m-n}-\frac{m(a-2n)}{a-x}=\frac{1-n^2}{n-m}-\frac{n(a+m)}{x-a},$$

or, $$\frac{m^2-1}{m-n}-\frac{n^2-1}{m-n}=\frac{am-2mn}{a-x}-\frac{-an-mn}{a-x};$$

or, $$m+n=\frac{am+an-mn}{a-x}.$$

Clearing of fractions, $$(m+n)(a-x)=am+an-mn.$$

Whence, $$-mx-nx=-mn,$$

and $$x=\frac{mn}{m+n}.$$

12. $$\frac{1}{x-a}-\frac{1}{x-a+c}=\frac{1}{x-b-c}-\frac{1}{x-b},$$

or, $$\frac{c}{(x-a)(x-a+c)}=\frac{c}{(x-b)(x-b-c)}.$$

Dividing numerators by c,

$$\frac{1}{(x-a)(x-a+c)}=\frac{1}{(x-b)(x-b-c)}.$$

Clearing of fractions,

$$x^2-2bx-cx+b^2+bc=x^2-2ax+cx+a^2-ac.$$

Whence, $$2(a-b-c)x=a^2-ac-bc-b^2,$$

and $$x=\frac{(a+b)(a-b-c)}{2(a-b-c)}=\frac{a+b}{2}.$$

13. $$\frac{x-a}{x-a-1}-\frac{x-a-1}{x-a-2}=\frac{x-b}{x-b-1}-\frac{x-b-1}{x-b-2},$$

or, $$\frac{1}{(x-a-1)(x-a-2)}=\frac{1}{(x-b-1)(x-b-2)}.$$

Clearing of fractions,

$$x^2-2bx-3x+b^2+3b+2=x^2-2ax-3x+a^2+3a+2.$$

Whence, $$2(a-b)x=(a+b+3)(a-b),$$

$$x=\frac{a+b+3}{2}.$$

14. $$(x+a)(2x+b+c)^2 = (x+b)(2x+a+c)^2.$$

Simplifying, $$(b^2+2ac-a^2-2bc)\,x = a^2b+bc^2-ab^2-ac^2,$$

or, $$(b-a)(a+b-2c)\,x = (c^2-ab)(b-a).$$

Whence, $$x = \frac{c^2-ab}{a+b-2c}.$$

16. $$(x-a)^3(x+a-2b) = (x-b)^3(x-2a+b).$$

Simplifying, $$2a^3x-2b^3x = 2ab^3-b^4-2a^3b+a^4.$$

Whence, $$x = \frac{a^4+2ab^3-2a^3b-b^4}{2\,(a^3-b^3)} = \frac{(a^3-b^3)(a+b)}{2\,(a^3-b^3)},$$

or, $$x = \frac{a+b}{2}.$$

17. $$\frac{3abc}{a+b} + \frac{a^2b^2}{(a+b)^3} + \frac{(2a+b)\,b^2x}{a\,(a+b)^2} = 3ax + \frac{bx}{a},$$

or, $$-\frac{(2a+b)\,b^2x}{a\,(a+b)^2} + \frac{3acx+bx}{a} = \frac{3abc}{a+b} + \frac{a^2b^2}{(a+b)^3}.$$

Simplifying,

$$\frac{3a^3cx+6a^2bcx+3ab^2cx+a^2bx}{a\,(a+b)^2} = \frac{b\,(3a^3c+6a^2bc+3ab^2c+a^2b)}{(a+b)^3}.$$

Clearing of fractions,

$$(a+b)(3a^3c+6a^2bc+3ab^2c+a^2b)\,x = ab\,(3a^3c+6a^2bc+3ab^2c+a^2b).$$

Whence, $$x = \frac{ab}{a+b}.$$

18. $$\tfrac{1}{3}(\tfrac{1}{6}x-2) - \frac{(x-3)(\tfrac{1}{4}x-10)}{9} = \tfrac{1}{6}(\tfrac{1}{6}x-2)(42-x).$$

Simplifying, $$\frac{-x^2+45x-144}{36} = \frac{54x-x^2-504}{36}.$$

Whence, $$9x = 360,$$

and $$x = 40.$$

19.
$$\frac{a^2(a+c)}{(a-b)(x-a)}-\frac{a^2(b+c)}{(a-b)(x-b)}=\frac{(a^3+a^2x)\,n}{x^2-a^2}.$$

Simplifying,
$$\frac{(a+c)}{(a-b)(x-a)}-\frac{(b+c)}{(a-b)(x-b)}=\frac{n}{x-a}.$$

Clearing of fractions,
$$(a+c)(x-b)-(b+c)(x-a)=n(x-b)(a-b).$$

Whence, $$ax-bc-bx+ac=n(x-b)(a-b),$$

or, $$(a-b)\,x+c(a-b)=n(x-b)(a-b).$$

Dividing by $(a-b)$, $$x+c=nx-nb.$$

Whence, $$x-nx=-c-nb,$$

or, $$x=\frac{bn+c}{n-1}.$$

20.
$$5x+1-\left\{13+6\left[2x-\frac{3(x+3)}{23}+\frac{1}{5}\left(\frac{x+10}{6}\right)\right]\right\}$$
$$=\left(6-\frac{x}{2}\right)\left(\frac{x}{2}+5\right)+\left(\frac{x}{4}-9\right)x.$$

Simplifying, $$\frac{-738x-1340}{115}=\frac{60-17x}{2}.$$

Whence, $$479x=9580,$$

and $$x=20.$$

21.
$$\frac{5(\frac{1}{4}+x)}{1+3x}+\frac{17-x}{2x+1}=18\tfrac{1}{4}-\frac{10x}{1+3x},$$

or, $$\frac{5+4x}{4(1+3x)}+\frac{17-x}{2x+1}=\frac{73}{4}-\frac{10x}{1+3x}.$$

Transposing, $$\frac{17-x}{2x+1}=\frac{73}{4}-\frac{10x}{1+3x}-\frac{5+4x}{4(1+3x)}.$$

Uniting, $$\frac{17-x}{2x+1} = \frac{68+159x}{4(1+3x)}.$$

Clearing of fractions and uniting,

$$330x^2 = -95x.$$

Whence, $$x = -\tfrac{19}{66}.$$

22. $$\frac{3(m+n)^2}{cm^2n^2} + (m+n) = \left[\frac{3(m+n)}{mn}\right]\left(\frac{m+n}{cmn}\right) - mn + (x+1)(m+n).$$

Simplifying, $$m+n = -mn+(m+n)x+m+n.$$

Whence, $$-(m+n)x = -mn.$$

$$x = \frac{mn}{m+n}.$$

23. $$(x-a)(x-b)(x+2a+2b) = (x+2a)(x+2b)(x-a-b).$$

Simplifying, $$3abx = 6a^2b+6ab^2.$$

$$x = 2(a+b).$$

Art. 410.

1. Let x = number of pounds of copper to be added.

Then $$\frac{100(140+x)}{200+x} = 84.$$

Whence, $$16x = 2800,$$

and $$x = 175.$$

2. Let x = number of cwt. of speculum metal needed.

Then $$\tfrac{9}{10}(8)+\tfrac{67}{100}(x) = \tfrac{4}{5}(8+x).$$

Whence, $$8x = 120,$$

and $$x = 15.$$

3. Let $x =$ number of oz. of first bar needed,

and $20-x =$ number of oz. of second bar needed.

Then $\frac{7}{10}x+\frac{4}{10}(20-x) = 10.$

Whence, $3x = 20,$

and $x = 6\frac{2}{3}.$

Hence, $20-x = 13\frac{1}{3}.$

4. Let $x =$ number of minute-spaces the minute-hand must go after 2 o'clock to overtake the hour-hand.

Then 12 gone : x gone :: 11 gained : 10 gained.

Whence, $11x = 120,$

and $x = 10\frac{10}{11}.$

5. Let $x =$ number of minute-spaces the minute-hand must go after 6 o'clock to be five minute-spaces behind the hour-hand.

Then 12 gone : x gone :: 11 gained : 25 gained.

Whence, $11x = 300,$

and $x = 27\frac{3}{11}.$

Let $x =$ number of minute-spaces the minute-hand must pass over to be ten minute-spaces ahead of the hour-hand.

Then 12 gone : x gone :: 11 gained : 40.

Whence, $11x = 480,$

and $x = 43\frac{7}{11}.$

6. Let $x =$ the minute-spaces the hour-hand must go,

and $12x =$ the minute-spaces the minute-hand must go.

Then $(45+x)-12x = 30.$

Whence, $11x = 15,$

and $x = 1\frac{4}{11}.$

Hence, $12x = 16\frac{4}{11}.$

7. Let $x =$ the minute-spaces the hour-hand must go,

and $12x =$ the minute-spaces the minute hand must go.

Then $12x-(35+x) = 15.$

Whence, $11x = 50,$

and $x = 4\frac{6}{11}.$

Hence, $12x = 54\frac{6}{11}.$

8. Let $x =$ the minute-spaces the hour-hand must go,

and $12x =$ the minute-spaces the minute-hand must go.

Then $(55+x)-12x = 30.$

Whence, $11x = 25,$

and $x = \frac{25}{11}.$

Hence, $12x = 27\frac{3}{11}.$

9. Let $x =$ the minute-spaces the hour-hand must go,

and $12x =$ the minute-spaces the minute-hand must go.

Then $12x+x = 35.$

Whence, $13x = 35,$

and $x = 2\frac{9}{13}.$

$35-2\frac{9}{13} = 32\frac{4}{13}$ min. after 7 o'clock.

10. Let $x =$ the minute-spaces the hour-hand must go,

and $12x =$ the minute-spaces the minute-hand must go.

Then $12x+x = 30.$

Whence, $13x = 30,$

and $x = 2\frac{4}{13}.$

Hence, $12x = 27\frac{9}{13}$ min. before 6.

$60-27\frac{9}{13} = 32\frac{4}{13}$ min. after 5.

11. Let $x =$ the minute-spaces the hour-hand must go,

and $12x =$ the minute-spaces the minute-hand must go.

Then $12x+x = 60-40.$

Whence, $x = 1\frac{7}{13}.$

Hence, $12x = 18\frac{6}{13}.$

12. Let $x =$ number of dollars that A has,

and $x-175 =$ number of dollars that B has.

Then $\frac{2}{3}x = \frac{3}{4}(x-175).$

Whence, $8x = 9x-1575,$

and $x = 1575.$

Hence, $x-175 = 1400.$

13. Let $x =$ number of dollars that house cost,

and $750-x =$ number of dollars that carriage cost.

Then $1\frac{1}{2}x = 750-x.$

Whence, $3x = 1500-2x,$

and $x = 300.$

Hence, $750-300 = 450.$

14. Let $x =$ number of cents per bushel of potatoes,

and $x+10 =$ number of cents per bushel of corn.

Then $80x+100(x+.10) = 136.$

Whence, $180x = 126,$

and $x = .70.$

Hence, $x+10 = .80.$

15. Let $\frac{x}{4} =$ number of miles A walks per hour,

and $\frac{x}{5} =$ number of miles B walks per hour.

Then $12\left(\frac{x}{4}\right) - 9\left(\frac{x}{5}\right) = 24.$

Whence, $60x-36x = 480,$

and $x = 20.$

Hence, $\frac{x}{4} = 5,$

and $\frac{x}{5} = 4.$

16. Let $x =$ number of miles officer goes per hour.

Then $$\frac{100}{x} = \frac{100-6\,(1\tfrac{2}{3})}{6}.$$

Whence, $$90x = 600,$$

and $$x = 6\tfrac{2}{3}.$$

17. Let $x =$ No. mi. the boat goes per hour in still water,

and $\frac{1}{4}x =$ No. mi. the current goes per hour.

Then $$\frac{100}{x-\frac{1}{4}x} - \frac{100}{x+\frac{1}{4}x} = 6\tfrac{2}{3}.$$

Whence, $$\frac{400}{3x} - \frac{400}{5x} = \frac{20}{3},$$

$$2000-1200 = 100x,$$

and $$x = 8.$$

Hence, $$\tfrac{1}{4}x = 2.$$

18. Let $x =$ number of gallons of water needed.

Then $$\frac{92+x}{100+x} = \frac{61}{65}.$$

Whence, $$4x = 120,$$

and $$x = 30.$$

19. Let $x =$ No. mi. the boat goes per hour in still water,

and $y =$ No. mi. the current goes per hour.

Then $\dfrac{39}{x+y} = 2\tfrac{3}{5}$, or $x+y = 15;$ (1)

and $\dfrac{39}{x-y} = 4\tfrac{1}{3}$, or $x-y = 9.$ (2)

Adding (1) and (2), $$2x = 24,$$

and $$x = 12.$$

Whence, $$y = 3.$$

20. Let x = number of minute-spaces the hour-hand must go,

and $12x$ = number of minute-spaces the minute-hand must go.

Then $$x+12x+15 = 60,$$

$$13x = 45,$$

and $$x = 3\tfrac{6}{13}.$$

Hence, $$12x = 41\tfrac{7}{13} \text{ minutes after 3.}$$

$$60-41\tfrac{7}{13} = 18\tfrac{6}{13} \text{ minutes of 4.}$$

21. Let $3x$ = number of thief's steps,

and $2x$ = number of officer's steps.

$$\frac{3x+20}{8} = \frac{2x}{5}.$$

Whence, $$x = 100.$$

Hence, $$3x = 300, \quad \text{and} \quad 2x = 200.$$

22. Let x = No. dollars in his stock at first.

Then $1\tfrac{1}{3}(x-500)$ = No. dollars in his stock at end of first year;

$1\tfrac{1}{3}[1\tfrac{1}{3}(x-500)-500]$ = No. dollars in his stock at end of second year;

and $1\tfrac{1}{3}\{1\tfrac{1}{3}[1\tfrac{1}{3}(x-500)-500]-500\}$ = No. dollars in his stock at end of third year.

Then $$1\tfrac{1}{3}\{1\tfrac{1}{3}[1\tfrac{1}{3}(x-500)-500]-500\} = 2x.$$

Whence, $$\frac{64x-74000}{27} = 2x,$$

and $$x = 7400.$$

23. Let x = number of minutes the clock loses,

and $x+2$ = number of minutes the watch gains.

Then $$\frac{819}{60+(x+2)} = \frac{767}{60-x}.$$

Whence, $$1586x = 1586,$$

and $$x = 1.$$

Hence, $$x+2 = 3.$$

24. Let x = number of pounds of silver needed.

Then $$.30 \text{ of } 140 = 42 \text{ lbs. of gold.}$$

$$140-42 = 98 \text{ lbs. of silver.}$$

$$\frac{98+x}{140+x} = .75.$$

Whence, $$.25x = 7,$$

and $$x = 28.$$

25. Let x = number of dollars he had at first.

Then $$\tfrac{4}{7}x-(\tfrac{12}{56}x-10) = \text{No. dollars not stolen.}$$

$$\tfrac{4}{7}x-(\tfrac{12}{56}x-10) = \frac{5x+140}{14}.$$

$$\frac{5x+140}{14}-\left(\frac{15x+420}{140}-13\right) = \text{No. dollars left after paying carfare.}$$

$$\frac{5x+140}{14}-\left(\frac{15x+420}{140}-13\right) = \frac{5x+400}{20}.$$

$$\frac{5x+400}{20}-\left(\frac{x+80}{10}+14\right) = \tfrac{1}{8}(\tfrac{8}{7}x).$$

Whence, $$21x-280 = 20x,$$

and $$x = 280.$$

Art. 413.

1. Let $$x = \text{what youngest received.}$$

Then $$x+m = \text{what second received,}$$

$$x+2m = \text{what third received,}$$

and $$x+3m = \text{what fourth received.}$$

Hence, $$4x+6m = d.$$

Whence, $$x = \frac{d-6m}{4},$$

and $$x+3m = \frac{d-6m}{4}+3m = \frac{d+6m}{4}.$$

2. Let $x =$ number of sheep.

Then $\frac{a}{x} =$ the price the first paid,

and $\frac{b}{x} =$ the price the second paid.

$$\frac{a}{x} - c = \frac{b}{x}.$$

Whence, $$x = \frac{a-b}{c}.$$

Hence, $$\frac{a}{x} = \frac{ac}{a-b},$$

and $$\frac{b}{x} = \frac{bc}{a-b}.$$

3. Let $x =$ first part, $y =$ second part,

$z =$ third part, and $u =$ fourth part.

$$x+y+z+u = n. \quad (1)$$

$$x+r = y-r. \quad (2)$$

$$x+r = rz. \quad (3)$$

$$x+r = \frac{u}{r}. \quad (4)$$

Adding (2) and (1), $2x+z+u = n-2r$. (5)

Multiplying (5) by r, $2rx+rz+ur = nr-2r^2$. (6)

Adding (3) and (6), $(2r+1)x+ur = nr-2r^2-r$. (7)

Adding (7) and (4), $(r^2+2r+1)x = nr-2r^2-r-r^3$.

Whence, $$x = \frac{nr-r(r+1)^2}{(r+1)^2}.$$

Hence, $$y = \frac{nr+r(r+1)^2}{(r+1)^2},$$

$$z = \frac{n}{(r+1)^2},$$

and $$u = \frac{nr^2}{(r+1)^2}.$$

4. Let $x =$ value of first horse,

and $y =$ value of second horse.

Then
$$x+a = my, \tag{1}$$
and
$$y+a = nx. \tag{2}$$
Multiplying (1) by n,
$$nx-mny = -an. \tag{3}$$
Adding (3) and (2),
$$y-mny = -a-an.$$
Whence,
$$y = \frac{a(n+1)}{mn-1}.$$
Hence,
$$x = \frac{a(m+1)}{mn-1}.$$

5. Let $x =$ the distance that he went.

Then
$$\frac{x}{m}+\frac{x}{n} = h.$$
$$(m+n)x = hmn.$$
Hence,
$$x = \frac{hmn}{m+n}.$$

6. Let $x =$ the breadth of field, and $ax =$ the length of field.

Then
$$\frac{(x+c)(ax+b)}{160} = \frac{x \times ax}{160} + d.$$
Whence,
$$ax^2+acx+bx+bc = ax^2+160d,$$
and
$$(ac+b)x = 160d-bc.$$
Hence,
$$x = \frac{160d-bc}{ac+b},$$
and
$$ax = \frac{a(160d-bc)}{ac+b}.$$

7. Let $x =$ number of gallons each contained at first.

Then
$$x-b = m(x-a).$$
Whence,
$$x = \frac{am-b}{m-1}.$$

8. Let x = number of pounds of tea needed.

Then $$\frac{ax+bc}{d} = x+b.$$

Whence, $$ax+bc = dx+bd,$$

and $$(a-d)\,x = bd-bc.$$

Hence, $$x = \frac{b\,(d-c)}{a-d}.$$

9. Let x = number of days that he worked,

and $a-x$ = number of days he is idle.

Then $$bx-c\,(a-x) = d.$$

Whence, $$x = \frac{ac+d}{b+c}.$$

10. Let x = number of men in the reinforcement.

Then $$m+x : m :: d-e : r.$$

Whence, $$rm+rx = dm-em.$$

Hence, $$x = \frac{m\,(d-e-r)}{r}.$$

11. Let x = A's share, y = B's share,

z = C's share, and u = D's share.

Then $x+y+z+u = a$, or, $x+y+z+u = a.$ (1)

$mx = y$, or, $mx-y = 0.$ (2)

$z = x+y$, or, $z-x-y = 0.$ (3)

$u = y+z$, or, $u-y-z = 0.$ (4)

Adding (1) and (2), $(m+1)\,x+z+u = a.$ (5)

Subtracting (4) from (3), $2z-x-u = 0.$ (6)

Multiplying (5) by 2, $(2m+2)\,x+2z+2u = 2a.$ (7)

Subtracting (6) from (7), $(2m+3)\,x+3u = 2a.$ (8)

Adding (1) and (4), $x+2u = a.$ (9)

Multiplying (8) by 2, $(4m+6)\,x+6u = 4a.$ (10)

Multiplying (9) by 3, $$3x+6u = 3a. \tag{11}$$

Subtracting (11) from (10), $$(4m+3)\,x = a.$$

Whence, $$x = \frac{a}{4m+3}.$$

Hence, $$y = \frac{am}{4m+3}, \qquad z = \frac{a\,(m+1)}{4m+3}, \qquad u = \frac{a\,(2m+1)}{4m+3}.$$

12. Let x = how long it will take A to do it,

y = how long it will take B to do it,

z = how long it will take C to do it,

and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ = how long it will take all together to do it.

Then $$\frac{1}{x}+\frac{1}{y}=\frac{1}{a}, \tag{1}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b}, \tag{2}$$

$$\frac{1}{z}+\frac{1}{y}=\frac{1}{c}. \tag{3}$$

Subtracting (1) from (2), $$\frac{1}{y}-\frac{1}{z}=\frac{1}{a}-\frac{1}{b}. \tag{4}$$

Adding (3) and (4), $$\frac{2}{y}=\frac{1}{a}-\frac{1}{b}+\frac{1}{c}.$$

Whence, $$bcy-acy+aby = 2abc,$$

and $$y = \frac{2abc}{ab+bc-ac}.$$

Hence, $$x = \frac{2abc}{ac+bc-ab},$$

$$z = \frac{2abc}{ab+ac-bc}.$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{2abc}{ab+ac+bc}.$$

13. Let $x =$ number of ounces of gold in the mixture,

and $c-x =$ number of ounces of silver in the mixture.

Then $$\frac{x}{a}+\frac{c-x}{b}=d.$$

Whence, $$bx+ac-ax=abd.$$

Hence, $$x=\frac{a(c-bd)}{a-b}.$$

Art. 414.

29. $$\frac{\sqrt{x}}{\sqrt{a}}+\frac{\sqrt{a}}{\sqrt{x}}=\frac{\sqrt{a}}{\sqrt{b}}.$$

Squaring, $$\frac{x}{a}+2+\frac{a}{x}=\frac{a}{b},$$

or, $$\frac{x}{a}+\frac{a}{x}=\frac{a}{b}-2.$$

Reducing, $$\frac{x^2+a^2}{ax}=\frac{a-2b}{b}.$$

Clearing of fractions, transposing, and uniting,

$$bx^2-a(a-2b)x=-a^2b.$$

Whence, $$x=a(a-2b)\pm\sqrt{4a^2b^2+a^2(-4a^2b+4a^2b^2)},$$

or, $$x=\frac{a^2-2ab\pm a\sqrt{a^2-4ab}}{2b}=\frac{a^2\pm a\sqrt{a^2-4ab}}{2b}-a.$$

30. $$\frac{x}{\sqrt{x}+\sqrt{a-x}}+\frac{x}{\sqrt{x}-\sqrt{a-x}}=\frac{b}{\sqrt{x}}.$$

Adding first two members, $$\frac{2x\sqrt{x}}{2x-a}=\frac{b}{\sqrt{x}}.$$

Clearing of fractions, $$2x^2=2bx-ab,$$

or, $$2x^2-2bx=-ab.$$

Multiplying by 2, $$4x^2-4bx = -2ab.$$

Adding b^2 to both sides, $$4x^2-4bx+b^2 = b^2-2ab.$$

Extracting square root, $$2x-b = \pm\sqrt{b^2-2ab}.$$

Whence, $$x = \frac{b \pm \sqrt{b^2-2ab}}{2}$$

31. $$\frac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}} = \frac{b^2}{1}.$$

From Art. 300, (4), $$\frac{\sqrt{2ax+x^2}}{a+x} = \frac{b^2+1}{b^2-1}.$$

Squaring, $$\frac{2ax+x^2}{a^2+2ax+x^2} = \frac{b^4+2b^2+1}{b^4-2b^2+1}.$$

From Art. 300, (6), $$\frac{a^2}{(a+x)^2} = \frac{4b^2}{(b^2+1)^2}.$$

Extracting square root, $$\frac{a}{a-x} = \pm\frac{2b}{b^2+1}.$$

Clearing of fractions, $$2bx+2ab = ab^2+a.$$

Whence, $$2bx = ab^2-2ab+a,$$

or, $$x = \pm\frac{a(b-1)^2}{2b}.$$

32. $$\frac{\sqrt{\frac{a+x}{a-x}}+b}{\sqrt{\frac{a+x}{a-x}}-b} = \frac{3}{2}.$$

From Art. 300, (7), $$\frac{b}{\sqrt{\frac{a+x}{a-x}}} = \frac{1}{5}.$$

Squaring, $$\frac{b^2(a-x)}{a+x} = \frac{1}{25}.$$

Clearing of fractions, $$25ab^2-25b^2x = a+x.$$

Transposing and uniting, $$25b^2x+x=a(25b^2-1).$$

Whence, $$x=\frac{a(25b^2-1)}{25b^2+1}.$$

33. $$\frac{\sqrt{3x+1}+3}{\sqrt{3x+1}-3}=\frac{\sqrt{7x+8}+4\frac{4}{5}}{\sqrt{7x+8}-4\frac{4}{5}}.$$

From Art. 300, (7), $$\frac{\sqrt{3x+1}}{3}=\frac{\sqrt{7x+8}}{\frac{24}{5}}.$$

Squaring, $$\frac{3x+1}{9}=\frac{25(7x+8)}{576}.$$

Clearing of fractions, $$64(3x+1)=25(7x+8).$$

Transposing and uniting, $$17x=136,$$

$$x=8.$$

34. $$\frac{\sqrt{3x+1}+\sqrt{2x-7}}{\sqrt{3x+1}-\sqrt{2x-7}}=\frac{\sqrt{28x+1}+\sqrt{3(4x-5)}}{\sqrt{28x+1}-\sqrt{3(4x-5)}}.$$

From Art. 300, (7), $$\frac{\sqrt{3x+1}}{\sqrt{2x-7}}=\frac{\sqrt{28x+1}}{\sqrt{12x-15}}.$$

Squaring, $$\frac{3x+1}{2x-7}=\frac{28x+1}{12x-15}.$$

Clearing of fractions, $$36x^2-33x-15=56x^2-194x-7.$$

Transposing and uniting, $$20x^2-161x=-8.$$

Multiplying by 5, $$100x^2-805x=-40.$$

Adding $(\frac{161}{4})^2$, $$100x^2-805x+(\tfrac{161}{4})^2=\tfrac{25281}{16}.$$

Extracting square root, $$10x-\tfrac{161}{4}=\pm\tfrac{159}{4}.$$

Whence, $$10x=80 \text{ or } \tfrac{1}{2},$$

and $$x=8 \text{ or } \tfrac{1}{20}.$$

35. $$\frac{\sqrt{6x+1}+\sqrt{2x}}{\sqrt{6x+1}-\sqrt{2x}}=\frac{7\sqrt{x-4}+\sqrt{7x+8}}{7\sqrt{x-4}-\sqrt{7x+8}}.$$

Whence, $$\frac{\sqrt{6x+1}}{\sqrt{2x}}=\frac{7\sqrt{x-4}}{\sqrt{7x+8}}.$$

Squaring, $$\frac{6x+1}{2x} = \frac{49x-196}{7x+8}.$$

Clearing of fractions, $$42x^2+55x+8 = 98x^2-392x.$$

Transposing and uniting, $$56x^2-447x = 8.$$

Completing square and extracting square root,

$$112x-447 = \pm 449.$$

Whence, $$x = 8 \text{ or } -\tfrac{1}{56}.$$

36. $$\frac{\sqrt{3(x+1)}+\sqrt{x-2}}{\sqrt{3(x+1)}-\sqrt{x-2}} = \frac{\sqrt{9x+1}+\sqrt{2x+3}}{\sqrt{9x+1}-\sqrt{2x+3}}.$$

From this, $$\frac{\sqrt{3(x+1)}}{\sqrt{x-2}} = \frac{\sqrt{9x+1}}{\sqrt{2x+3}}.$$

Squaring, $$\frac{3x+3}{x-2} = \frac{9x+1}{2x+3}.$$

Clearing of fractions, $$6x^2+15x+9 = 9x^2-17x-2.$$

Transposing and uniting, $$3x^2-32x = 11.$$

Multiplying by 3, $$9x^2-96x = 33.$$

Adding 16^2, $$9x^2-96x+16^2 = 289.$$

Extracting square root, $$3x-16 = \pm 17.$$

Whence, $$x = 11 \text{ or } -\tfrac{1}{3}.$$

37. $$\frac{\sqrt{4x+5}+\sqrt{2x+3}}{\sqrt{4x+5}-\sqrt{2x+3}} = \frac{\sqrt{5x-6}+\sqrt{3x-8}}{\sqrt{5x-6}-\sqrt{3x-8}}.$$

From this, $$\frac{\sqrt{4x+5}}{\sqrt{2x+3}} = \frac{\sqrt{5x-6}}{\sqrt{3x-8}}.$$

Squaring, $$\frac{4x+5}{2x+3} = \frac{5x-6}{3x-8}.$$

Clearing of fractions, $$12x^2-17x-40 = 10x^2+3x-18.$$

Transposing and uniting, $2x^2-20x = 22,$

or, $x^2-10x = 11.$

Adding 5^2, $x^2-10x+25 = 36.$

Extracting square root, $x-5 = \pm 6.$

Whence, $x = 11 \text{ or } -1.$

38.
$$\frac{\sqrt{a+x}+\sqrt{b-x}}{\sqrt{a+x}-\sqrt{b-x}} = \frac{\sqrt{c+x}+\sqrt{a-x}}{\sqrt{c+x}-\sqrt{a-x}}.$$

From this, $$\frac{\sqrt{a+x}}{\sqrt{b-x}} = \frac{\sqrt{c+x}}{\sqrt{a-x}}.$$

Squaring, $$\frac{a+x}{b-x} = \frac{c+x}{a-x}.$$

Clearing of fractions, $a^2-x^2 = bc+(b-c)\,x-x^2.$

Transposing and uniting, $(b-c)\,x = a^2-bc.$

Whence, $$x = \frac{a^2-bc}{b-c}.$$

Art. 415.

28. $8\,(xy+1) = 33y;$ (1)

$4\,(xy+1) = 33x.$ (2)

Multiplying (2) by 2, $8\,(xy+1) = 66x.$ (3)

Subtracting (3) from (1), $33y-66x = 0,$

or, $y = 2x.$ (4)

Substituting value of y in (2),

$$4\,(2x^2+1) = 33x.$$

Transposing and uniting, $8x^2-33x = -4.$

Whence, $$x = \frac{33 \pm \sqrt{4\times 8\times(-4)+33^2}}{16},$$

or, $x = 4 \text{ or } \frac{1}{8}.$

Hence, $y = 8 \text{ or } \frac{1}{4}.$

29. $x^3-y^3 = 7930$; (1) $\qquad x-y = 10.$ (2)

Dividing (1) by (2), $\qquad x^2+xy+y^2 = 793.$ (3)

Squaring (2), $\qquad x^2-2xy+y^2 = 100.$ (4)

Subtracting (4) from (3), $\qquad 3xy = 693,$

or, $\qquad xy = 231.$ (5)

Adding (5) and (3), $\qquad x^2+2xy+y^2 = 1024.$

Extracting square root, $\qquad x+y = \pm 32.$ (6)

Adding (6) and (2), $\qquad 2x = 42$ or $-22,$

and $\qquad x = 21$ or $-11.$

Hence, $\qquad y = 11$ or $-21.$

30. $x^2(x-y) = 75$; (1) $\qquad x^2(2x+3y) = 400.$ (2)

Uniting, $\qquad \dfrac{2x+3y}{x-y} = \dfrac{16}{3},$

or, $\qquad 6x+9y = 16x-16y.$

Transposing and uniting, $\qquad 10x = 25y,$

or, $\qquad x = \dfrac{5y}{2}.$

Substituting value of x in (2), $\qquad \dfrac{25y^2}{4}(5y+3y) = 400.$

Whence, $\qquad 200y^3 = 1600, \qquad y^3 = 8,$ and $y = 2.$

Substituting value of x in (1), $\qquad \dfrac{25y^2}{4} \times \dfrac{3y}{2} = 75.$

Whence, $\qquad 75y^3 = 600,$

$$y^3 = 8,$$

and $\qquad y^3-8 = 0.$

Factoring, $\qquad (y-2)(y^2+2y+4) = 0.$

$$y^2+2y+4 = 0, \quad \text{or,} \quad y^2+2y = -4.$$

Extracting square root, $\qquad y = -1 \pm \sqrt{-3}.$

See Appendix.

31. $x^2-xy+y^2 = 49-2xy$; (1) $x^4+x^2y^2+y^4 = 931.$ (2)

Or, $x^2+xy+y^2 = 49$; (1) $x^4+x^2y^2+y^4 = 931.$ (2)

Dividing (2) by (1), $x^2-xy+y^2 = 19.$ (3)

Subtracting (3) from (1), $2xy = 30,$

or, $xy = 15.$ (4)

Subtracting (4) from (3), $x^2-2xy+y^2 = 4.$

Extracting square root, $x-y = \pm 2.$ (5)

Adding (4) and (1), $x^2+2xy+y^2 = 64.$

Extracting square root, $x+y = \pm 8.$ (6)

Adding (5) and (6), $2x = \pm 10$ or $\pm 6,$

or, $x = \pm 5$ or $\pm 3.$

Hence, $y = \pm 3$ or $\pm 5.$

32. $x^2+y^2+4(x^2+y^2)^{\frac{1}{2}} = 45$; (1) $x^4+y^4 = 337.$ (2)

$$x^2+y^2+4\sqrt{x^2+y^2} = 45.$$

Adding 4 to both sides, $x^2+y^2+4\sqrt{x^2+y^2}+4 = 49.$

Extracting square root, $\sqrt{x^2+y^2}+2 = \pm 7,$

or, $\sqrt{x^2+y^2} = 5$ or $-9.$

Squaring, $x^2+y^2 = 25$ or $81.$ (3)

Squaring, $x^4+2x^2y^2+y^4 = 625.$ (4)

Subtracting (4) from (2), $-2x^2y^2 = -288.$ (5)

Adding (5) and (2), $x^4-2x^2y^2+y^4 = 49.$

Extracting square root, $x^2-y^2 = \pm 7.$ (6)

Adding (3) and (6), $2x^2 = 32$ or $18.$

$$x^2 = 16 \text{ or } 9.$$

Extracting square root, $x = \pm 4$ or $\pm 3.$

Hence, $y = \pm 3$ or $\pm 4.$

33. $x^2+2xy+y^2+2x = 120-2y$; (1) $\qquad xy-y^2 = 8.$ (2)

Or, $(x+y)^2+2(x+y) = 120$; (1) $\qquad xy-y^2 = 8.$ (2)

From (1),
$$x+y = -1 \pm \sqrt{121}$$
$$= -12 \text{ or } 10.$$

Whence,
$$x = 10-y \text{ or } -12-y.$$

Substituting value of x in (2),
$$y(10-y)-y^2 = 8, \tag{3}$$

or,
$$y(-12-y)-y^2 = 8. \tag{4}$$

Simplifying (3),
$$y^2-5y = -4. \tag{5}$$

Extracting square root, $\quad y-\frac{5}{2} = \pm\frac{3}{2}$, or, $y = 1$ or 4.

Simplifying (4),
$$y^2+6y = -4.$$

Extracting square root,
$$y = -3 \pm \sqrt{5}.$$

Whence,
$$x = 9,\ 6,\ \text{or } -9 \mp \sqrt{5}.$$

34. $3x^2+xy = 68$; (1) $\qquad 4y^2+3xy = 160.$ (2)

Let $x = ry$.

Then
$$\left.\begin{array}{l} 3x^2+\ xy = \ 68;\ (1) \\ 4y^2+3xy = 160;\ (2) \end{array}\right\} = \left\{\begin{array}{l} 3r^2y^2+\ ry^2 = \ 68.\ (3) \\ \ 4y^2+3ry^2 = 160.\ (4) \end{array}\right\}$$

From (3),
$$y^2 = \frac{68}{3r^2+r}. \tag{5}$$

From (4),
$$y^2 = \frac{160}{4+3r}. \tag{6}$$

Hence,
$$\frac{68}{3r^2+r} = \frac{160}{4+3r},$$

or,
$$120r^2-11r = 68.$$

Whence,
$$r = \frac{11 \pm \sqrt{4\times 68\times 120+11^2}}{240}$$
$$= \tfrac{4}{5} \text{ or } -\tfrac{17}{24}.$$

Substituting value of r in (6), $y^2 = \dfrac{160}{4+\frac{12}{5}}$ or $\dfrac{160}{4-\frac{51}{24}}$
$$= 25 \text{ or } \tfrac{256}{3}.$$

Extracting square root,
$$y = \pm 5 \text{ or } \pm \tfrac{16}{3}\sqrt{3}.$$

Hence,
$$x = \pm 4 \text{ or } \mp \tfrac{34}{9}\sqrt{3}.$$

35. $x^2y^4+y^2=333;\quad(1)\qquad xy^2+y=21.\quad(2)$

Squaring (2), $x^2y^4+2xy^3+y^2=441.\quad(3)$

Subtracting (3) from (1), $-2xy^3=-108.\quad(4)$

Adding (4) and (1), $x^2y^4-2xy^3+y^2=225.$

Extracting square root, $xy^2-y=\pm 15.\quad(5)$

Subtracting (5) from (2), $2y=\pm 36$ or $\pm 6,$

and $y=\pm 3$ or $\pm 18.$

Hence, $x=\pm 2$ or $\pm \frac{1}{108}.$

36. $x^4+x^2y^2+y^4=133;\quad(1)\qquad x^2-xy+y^2=7.\quad(2)$

Dividing (1) by (2), $x^2+xy+y^2=19.\quad(3)$

Subtracting (2) from (3), $2xy=12,$

and $xy=6.\quad(4)$

Adding (4) and (3), $x^2+2xy+y^2=25.$

Extracting square root, $x+y=\pm 5.\quad(5)$

Subtracting (4) from (2), $x^2-2xy+y^2=1.$

Extracting square root, $x-y=\pm 1.\quad(6)$

Adding (5) and (6), $2x=\pm 6$ or $\pm 4,$

and $x=\pm 3$ or $\pm 2.$

Hence, $y=\pm 2$ or $\pm 3.$

Art. 420.

1. $$7x-7\tfrac{2}{3} > \frac{2x}{3}+5.$$

Transposing, $7x-\dfrac{2x}{3} > 12\tfrac{2}{3}.$

Clearing of fractions, $21x-2x > 38,$

or, $19x > 38.$

Whence, $x > 2.$

2. $$x + \frac{x}{2} + \frac{x}{3} > 11.$$

Clearing of fractions, $6x+3x+2x > 66.$

Whence, $x > 6.$

3. $$\frac{x}{2} + 3x - 5 > 16.$$

Clearing of fractions, $x+6x-10 > 32.$

Transposing and uniting, $7x > 42.$

Whence, $x > 6.$

4. $$7x-1 > 34.$$

Transposing and uniting, $7x > 35.$

Whence, $x > 5.$

5. $$\frac{a^2-b^2}{6d} > \frac{a^2-b^2}{3x}.$$

Clearing of fractions $3(a^2-b^2)x > 6d(a^2-b^2).$

Whence, $x > 2d.$

8. $$\frac{ax}{5} + bx - ab > \frac{a^2}{5}.$$

Clearing of fractions, $ax+5bx-5ab > a^2.$

Transposing, $(a+5b)x > a(a+5b).$

Whence, $x > a.$

9. $$\frac{bx}{7} - ax + ab < \frac{b^2}{7}.$$

Clearing of fractions, $bx-7ax+7ab < b^2.$

Transposing, $(b-7a)x < b(b-7a).$

Whence, $x < b.$

10. $3x-4 < x+6$; (1) $\qquad 5x+7 > 3x+13.$ (2)

Transposing and uniting, we have from (1),

$2x < 10,$ or, $x < 5.$ (3)

From (2), $2x > 6,$ or, $x > 3.$

Hence, $x = 4.$

11.
$$\tfrac{1}{4}(x+2)+\tfrac{1}{3}x < \tfrac{1}{2}(x-4)+3; \qquad (1)$$
$$\tfrac{1}{4}(x+2)+\tfrac{1}{3}x > \tfrac{1}{2}(x+1)+\tfrac{1}{3}. \qquad (2)$$

Clearing of fractions in (1),

$3x+6+4x < 6x-24+36.$ Whence, $x < 6.$

Clearing of fractions in (2),

$3x+6+4x > 6x+6+4.$ Whence, $x > 4.$

Hence, $x = 5.$

12. Let $x =$ the number.

Then $3x+2 > 2x+61$; (1) $5x-70 < 4x-9.$ (2)

Transposing and uniting in (1), $x > 59.$

Transposing and uniting in (2), $x < 61.$

Hence, $x = 60.$

13. Let $x =$ greater, and $y =$ less.

Then $x > 3y$; (1) $y > \frac{x}{5}$; (2) and $x+y = 25.$ (3)

From (3), $x = 25-y.$

Substituting in (1) and in (2),

$y > 4\tfrac{1}{6}$ and $y < 6\tfrac{1}{4}.$

Whence, $y = 5$ or 6, and $x = 20$ or $19.$

Art. 426.

3. Let $x =$ the second root.

Since multiplying the two roots of the equation together produces q,

$$(a+\sqrt{a^2-b^2})\times x = q \text{ or } b^2.$$

Whence,
$$x = \frac{b^2}{a+\sqrt{a^2-b^2}}.$$

Multiplying both numerator and denominator by $a-\sqrt{a^2-b^2}$,

$$x = \frac{b^2(a-\sqrt{a^2-b^2})}{b^2} \text{ or } a-\sqrt{a^2-b^2}.$$

7. $$x^2+3x-10=0.$$

Factoring $$(x+5)(x-2)=0.$$

Whence, $$x+5=0, \qquad x=-5;$$

and $$x-2=0, \qquad x=2.$$

8. $$x^2+7x+12=0.$$

Factoring, $$(x+4)(x+3)=0.$$

Whence, $$x+4=0, \qquad x=-4;$$

and $$x+3=0, \qquad x=-3.$$

9. $$x^2-8x-65=0.$$

Factoring, $$(x-13)(x+5)=0.$$

Whence, $$x-13=0, \qquad x=13;$$

and $$x+5=0, \qquad x=-5.$$

10. $$5x^2+11x+2=0.$$

Factoring, $$(5x+1)(x+2)=0.$$

Whence, $$5x+1=0, \qquad x=-\tfrac{1}{5};$$

and $$x+2=0, \qquad x=-2.$$

11. $$2x^2+11x+15=0.$$

Factoring, $$(x+3)(2x+5)=0.$$

Whence, $$x+3=0, \qquad x=-3;$$

and $$2x+5=0, \qquad x=-2\tfrac{1}{2}.$$

12. $$39x^2-34x+7=0.$$

Factoring, $$(13x-7)(3x-1)=0.$$

Whence, $$13x-7=0, \qquad x=\tfrac{7}{13};$$

and $$3x-1=0, \qquad x=\tfrac{1}{3}.$$

13. $$21x^2+11x-2=0.$$

Factoring, $$(7x-1)(3x+2)=0.$$

Whence, $$7x-1=0, \qquad x=\tfrac{1}{7};$$

and $$3x+2=0, \qquad x=-\tfrac{2}{3}.$$

14. $35x^2-2x-117=0.$

Factoring, $(7x-13)(5x+9)=0.$

Whence, $7x-13=0,\quad x=1\frac{6}{7};$

and $5x+9=0,\quad x=-1\frac{4}{5}.$

15. $22x^2-41x-35=0.$

Factoring, $(11x+7)(2x-5)=0.$

Whence, $11x+7=0,\quad x=-\frac{7}{11};$

and $2x-5=0,\quad x=2\frac{1}{2}.$

16. $8x^2-19x-27=0.$

Factoring, $(x+1)(8x-27)=0.$

Whence, $x+1=0,\quad x=-1;$

and $8x-27=0,\quad x=3\frac{3}{8}.$

17. $x^3+x^2-17x+15=0.$

Factoring, $(x-3)(x+5)(x-1)=0.$

Whence, $x-3=0,\quad x=3;$

$x+5=0,\quad x=-5;$

and $x-1=0,\quad x=1.$

18. $x^3+10x^2+31x+30=0.$

Factoring, $(x+2)(x+3)(x+5)=0.$

Whence, $x+2=0,\quad x=-2;$

$x+3=0,\quad x=-3;$

and $x+5=0,\quad x=-5.$

19. $x^3-10x^2+23x-14=0.$

Factoring, $(x-1)(x-2)(x-7)=0.$

Whence, $x-1=0,\quad x=1;$

$x-2=0,\quad x=2;$

and $x-7=0,\quad x=7.$

20. $x^3+x^2-65x+63=0.$

Factoring, $(x-1)(x-7)(x+9)=0.$

Whence, $x-1=0, \quad x=1;$

$x-7=0, \quad x=7,$

and $x+9=0, \quad x=-9.$

21. $x^3+3x^2-10x-24=0.$

Factoring, $(x+2)(x-4)(x+8)=0.$

Whence, $x+2=0, \quad x=-2;$

$x-4=0, \quad x=4;$

and $x+8=0, \quad x=-8.$

22. $x^3+6x^2-24x-64=0.$

Factoring, $(x+2)(x-4)(x+8)=0.$

Whence, $x+2=0, \quad x=-2;$

$x-4=0, \quad x=4;$

and $x+8=0, \quad x=-8.$

23. $x^3-19x-30=0.$

Factoring, $(x+2)(x+3)(x-5)=0.$

Whence, $x+2=0, \quad x=-2;$

$x+3=0, \quad x=-3;$

and $x-5=0, \quad x=5.$

24. $x^3-6x^2-67x-60=0.$

Factoring, $(x+1)(x+5)(x-12)=0.$

Whence, $x+1=0, \quad x=-1;$

$x+5=0, \quad x=-5;$

and $x-12=0, \quad x=12.$

25. $3x^3-x^2-19x-15=0.$

Factoring, $(x+1)(x-3)(3x+5)=0.$

Whence, $x+1=0, \quad x=-1;$

$x-3=0, \quad x=3;$

and $3x+5=0, \quad x=-\frac{5}{3}.$

26. $$x^3-14x^2+35x-22=0.$$

Factoring, $$(x-11)(x-2)(x-1)=0.$$

Whence, $x-11=0, \quad x=11;$

$x-2=0, \quad x=2;$

and, $x-1=0, \quad x=1.$

Art. 433.

1. $$2x+3y=9. \qquad (1)$$

Whence, $$x=\frac{9-3y}{2} \text{ or } 4-y+\frac{1-y}{2}.$$

Since x is integral, $\frac{1-y}{2}$ must be integral or equal to zero.

Let $$\frac{1-y}{2}=m.$$

Then $$y=1-2m.$$

Substituting in (1), $$2x+3(1-2m)=9.$$

Whence, $$x=3+3m.$$

Making $m=0$, $x=3$, and $y=1.$

2. $$4x+29y=150. \qquad (1)$$

Whence, $$x=\frac{150-29y}{4} \text{ or } 37-7y+\frac{2-y}{4}.$$

Since x is integral, $\frac{2-y}{4}$ must be integral or equal to zero.

Let $$\frac{2-y}{4}=m.$$

Then $$y=2-4m.$$

Substituting in (1), $$x=23+29m.$$

Making $m=0$, $y=2$, and $x=23.$

3. $$3x+29y = 151. \qquad (1)$$

Whence, $$x = \frac{151-29y}{3} \text{ or } 50-9y+\frac{1-2y}{3}.$$

$$\left(\frac{1-2y}{3}\right)\times 2 = \frac{2-4y}{3} = -y-\frac{y-2}{3}.$$

Since x is integral, $\frac{y-2}{3}$ is integral or equal to zero.

Let $$\frac{y-2}{3} = m.$$

Then $$y = 2+3m.$$

Substituting in (1), $$3x+29(2+3m) = 151.$$

Whence, $$x = 31-29m.$$

Let $m = 1$. Then $$y = 5, \text{ and } x = 2.$$

4. $$7x+15y = 225. \qquad (1)$$

Whence, $$x = \frac{225-15y}{7} \text{ or } 32-2y+\frac{1-y}{7}.$$

Since x is integral, $\frac{1-y}{7}$ is integral or equal to zero.

Let $$\frac{1-y}{7} = m.$$

Then $$y = 1-7m.$$

Substituting in (1), $$7x+15(1-7m) = 225.$$

Whence, $$x = 30+15m.$$

Let $m = -1$. Then $$x = 15, \text{ and } y = 8.$$

5. $$3x+4y = 39.$$

Whence, $$x = \frac{39-4y}{3} \text{ or } 13-y-\frac{y}{3}.$$

Since x is integral, $\frac{y}{3}$ is integral or equal to zero.

Let $$\frac{y}{3} = m.$$

Then $$y = 3m.$$

Substituting in (1), $$3x+12m = 39.$$

Whence, $$x = 13-4m.$$

Let $m = 2$. Then $$x = 5, \quad \text{and} \quad y = 6.$$

6. $$8x+13y = 500. \quad (1)$$

Whence, $$x = \frac{500-13y}{8} \text{ or } 62-y+\frac{4-5y}{8}.$$

Since x is integral, $\frac{4-5y}{8}$ is integral or equal to zero.

$$\left(\frac{4-5y}{8}\right)\times 5 = \frac{20-25y}{8} \text{ or } 2-3y+\frac{4-y}{8}.$$

Let $$\frac{4-y}{8} = m.$$

Then $$y = 4-8m.$$

Substituting in (1), $$8x+13(4-8m) = 500.$$

Whence, $$x = 56+13m.$$

Let $m = -2$. Then $$x = 30, \quad \text{and} \quad y = 20.$$

7. $$7x+13y = 405. \quad (1)$$

Whence, $$x = \frac{405-13y}{7} \text{ or } 57-y+\frac{6-6y}{7}.$$

Since x is integral, $\frac{6-6y}{7}$ is integral or equal to zero.

$$\left(\frac{6-6y}{7}\right)\div 6 = \frac{1-y}{7}.$$

Let $$\frac{1-y}{7} = m.$$

Then $$y = 1-7m.$$

Substituting in (1), $$7x+13(1-7m) = 405.$$

Whence, $$x = 56+13m.$$

Let $m = -2$. Then $$x = 30, \quad \text{and} \quad y = 15.$$

8. $$2x+7y = 125. \qquad (1)$$

Whence, $$x = \frac{125-7y}{2} \text{ or } 62-3y+\frac{1-y}{2}.$$

Since x is integral, $\frac{1-y}{2}$ is integral or equal to zero.

Let $$\frac{1-y}{2} = m.$$

Then $$y = 1-2m.$$

Substituting in (1), $$2x+7(1-2m) = 125.$$

Whence, $$x = 59+7m.$$

Let $m = -6$. Then $$x = 17, \text{ and } y = 13.$$

9. $$19x-14y = 11. \qquad (1)$$

Whence, $$x = \frac{11+14y}{19}.$$

Since x is integral, $\frac{11+14y}{19}$ is integral or equal to zero.

$$\left(\frac{11+14y}{19}\right)\times 11 = \frac{121+154y}{19} = 6+8y+\frac{7+2y}{19}.$$

$$\left(\frac{7+2y}{19}\right)\times 10 = \frac{70+20y}{19} = 3+y+\frac{13+y}{19}.$$

Let $$\frac{13+y}{19} = m.$$

Then $$y = 19m-13.$$

Substituting in (1), $$19x-14(19m-13) = 11.$$

Whence, $$x = 14m-9.$$

Let $m = 1$. Then $$x = 5, \text{ and } y = 6.$$

10. $$17x-7y = 1. \qquad (1)$$

Whence, $$x = \frac{1+7y}{17}.$$

Since x is integral, $\frac{1+7y}{17}$ is integral or equal to zero.

$$\left(\frac{1+7y}{17}\right)\times 15=\frac{15+105y}{17}=6y+\frac{15+3y}{17}.$$

$$\left(\frac{15+3y}{17}\right)\times 6=\frac{90+18y}{17}=5+y+\frac{5+y}{17}.$$

Let $\frac{5+y}{17}=m.$

Then $y=17m-5.$

Substituting in (1), $17x-7(17m-5)=1.$

Whence, $x=7m-2.$

Let $m=1$. Then $x=5,$ and $y=12.$

11. $23x-9y=929.$ (1)

Whence, $x=\frac{929+9y}{23}$ or $40+\frac{9+9y}{23}.$

Since x is integral, $\frac{9+9y}{23}$ is integral or equal to zero.

$$\left(\frac{9+9y}{23}\right)\div 9=\frac{1+y}{23}.$$

Let $\frac{1+y}{23}=m.$

Then $y=23m-1.$

Substituting in (1), $23x-9(23m-1)=929.$

Whence, $x=9m+40.$

Let $m=1$. Then $x=49,$ and $y=22.$

12. $8x-23y=19.$ (1)

Whence, $x=\frac{19+23y}{8}$ or $2+2y+\frac{3+7y}{8}.$

Since x is integral, $\frac{3+7y}{8}$ is integral or equal to zero.

$$\left(\frac{3+7y}{8}\right)\times 7 = \frac{21+49y}{8} = 2+6y+\frac{5+y}{8}.$$

Let $\frac{5+y}{8} = m.$

Then $y = 8m-5.$

Substituting in (1), $8x-23(8m-5) = 19.$

Whence, $x = 23m-12.$

Let $m = 1$. Then $x = 11$, and $y = 8.$

13. $39x-56y = 11.$ (1)

Whence, $x = \frac{11+56y}{39}$ or $y+\frac{11+17y}{39}.$

Since x is integral, $\frac{11+17y}{19}$ is integral or equal to zero.

$$\left(\frac{11+17y}{39}\right)\times 7 = \frac{77+119y}{39} = 1+3y+\frac{38+2y}{39}.$$

$$\left(\frac{38+2y}{39}\right)\div 2 = \frac{19+y}{39}.$$

Let $\frac{19+y}{39} = m.$

Then $y = 39m-19.$

Substituting in (1), $39x-56(39m-19) = 11.$

Whence, $x = 56m-27.$

Let $m = 1$. Then $x = 29$, and $y = 20.$

14. $3x+5y = 73.$ (1)

Whence, $x = \frac{73-5y}{3}.$

Since x is integral, $\frac{73-5y}{3}$ is integral or equal to zero.

$$\left(\frac{73-5y}{3}\right)\times 2 = \frac{146-10y}{3} = 48-3y+\frac{2-y}{3}.$$

Let $\dfrac{2-y}{3} = m.$

Then $y = 2-3m.$

Substituting in (1), $3x+5(2-3m) = 73.$

Whence, $x = 21+5m.$

Let $m = -2.$ Then $x = 11,$ and $y = 8.$

15. $5x+9y = 37.$ (1)

Whence, $x = \dfrac{37-9y}{5}$ or $7-y+\dfrac{2-4y}{5}.$

Since x is integral, $\dfrac{2-4y}{5}$ is integral or equal to zero.

$$\left(\frac{2-4y}{5}\right)\times 4 = \frac{8-16y}{5} \text{ or } 1-3y+\frac{3-y}{5}.$$

Let $\dfrac{3-y}{5} = m.$

Then $y = 3-5m.$

Substituting in (1), $5x+9(3-5m) = 37.$

Whence, $x = 2+9m.$

Let $m = 0.$ Then $x = 2,$ and $y = 3.$

16. $8x+11y = 49.$ (1)

Whence, $x = \dfrac{49-11y}{8}$ or $6-y+\dfrac{1-3y}{8}.$

Since x is integral, $\dfrac{1-3y}{8}$ is integral or equal to zero.

$$\left(\frac{1-3y}{8}\right)\times 11 = \frac{11-33y}{8} \text{ or } 1-4y+\frac{3-y}{8}.$$

Let $\dfrac{3-y}{8} = m.$

Then $y = 3-8m.$

Substituting in (1), $8x+11(3-8m) = 49.$

Whence, $x = 2+11m.$

Let $m = 0.$ Then $x = 2,$ and $y = 3.$

17.
$$\begin{array}{lrcll}
(1)\times 2 = & 4x+10y+6z &=& 102, & (1)\\
(2)\times 3 = & 30x+\ \ 9y+6z &=& 360. & (2)\\
\text{Subtracting,} & 26x-y &=& 258. & (3)
\end{array}$$

Whence, $$x=\frac{258+y}{26} \text{ or } 9+\frac{24+y}{26}.$$

Since x is integral, $\frac{24+y}{26}$ is integral or equal to zero.

Let $$\frac{24+y}{26}=m.$$

Then $$y=26m-24.$$

Substituting in (3), $$26x-(26m-24)=258.$$

Whence, $$x=m+9.$$

Substituting values of x and y in (1),

$$2(m+9)+5(26m-24)+3z=51.$$

Whence $$z=51-44m.$$

Let $m=1$. Then $$x=10,\quad y=2,\quad \text{and}\quad z=7.$$

18.
$$\begin{array}{lrcll}
(1)\times 7 = & 21x+35y+49z &=& 3920. & (1)\\
(2) = & 9x+25y+49z &=& 2920. & (2)\\
\text{Subtracting,} & 12x+10y &=& 1000, & \\
\text{or,} & 6x+\ \ 5y &=& 500. & (3)
\end{array}$$

Whence, $$x=\frac{500-5y}{6} \text{ or } 83+\frac{2-5y}{6}.$$

Since x is integral, $\frac{2-5y}{6}$ is integral or equal to zero.

$$\left(\frac{2-5y}{6}\right)\times 5=\frac{10-25y}{6} \text{ or } 1-4y+\frac{4-y}{6}.$$

Let $$\frac{4-y}{6}=m.$$

Then $$y=4-6m. \qquad (4)$$

Substituting in (3), $$6x+5(4-6m)=500.$$

Whence, $$x=80+5m. \qquad (5)$$

Substituting values of x and y in (1),

$$3(80+5m)+5(4-6m)+7z = 560.$$

Whence, $$z = \frac{300+15m}{7} = 42+2m+\frac{6+m}{7}.$$

Let $\frac{6+m}{7} = n$; whence, $m = 7n-6$. (6)

Substituting (6) in (4) and in (5), and the resulting values of x and y in (1),

$$x = 50+35n; \quad y = 40-42n; \quad z = 30-15n.$$

When $n = 0$, $x = 50$, $y = 40$, $z = 30$.

19. $$2x+11y-3z = 35. \quad (1)$$
$$3x-2y+3z = 16. \quad (2)$$

Adding, $$5x+9y = 51. \quad (3)$$

Whence, $$x = \frac{51-9y}{5} \text{ or } 10-y+\frac{1-4y}{5}.$$

Since x is integral, $\frac{1-4y}{5}$ is integral or equal to zero.

$$\left(\frac{1-4y}{5}\right)\times 4 = \frac{4-16y}{5} \text{ or } 3y+\frac{4-y}{5}.$$

Let $$\frac{4-y}{5} = m.$$

Then $$y = 4-5m.$$

Substituting in (3), $$5x+9(4-5m) = 51.$$

Whence, $$x = 3+9m.$$

Substituting values of x and y in (2),

$$3(3+9m)-2(4-5m)+3z = 16.$$

Whence, $$z = \frac{15-17m}{3} = 5-5m-\frac{2m}{3}.$$

Let $\frac{m}{3} = n$; whence, $m = 3n$.

Substituting as in Example 18,

$$x = 3+27n; \quad y = 4-15n; \quad z = 5-37n.$$

When $n = 0$, $x = 3$, $y = 4$, $z = 5$.

Art. 434.

1. Let x = number of calves, and y = number of hogs.

Then $$11x+13y = 131.$$

Whence, $$x = \frac{131-13y}{11} \text{ or } 11-y+\frac{10-2y}{11}.$$

$$\frac{10-2y}{11}\times 6 = \frac{60-12y}{11} \text{ or } 5-y+\frac{5-y}{11}.$$

Let $$\frac{5-y}{11} = m.$$

Then $$y = 5-11m.$$

Substituting in (1), $$11x+13(5-11m) = 131.$$

Whence, $$x = 6+13m.$$

Let $m = 0$. Then $x = 6$, and $y = 5$.

2. Let x = number to whom \$.70 was given,

and y = number to whom \$.30 was given.

Then $$70x+30y = 500,$$

or, $$7x+3y = 50. \quad (1)$$

Whence, $$x = \frac{50-3y}{7} \text{ or } 7+\frac{1-3y}{7}.$$

$$\frac{1-3y}{7}\times 5 = \frac{5-15y}{7} \text{ or } 2y+\frac{5-y}{7}.$$

Let $$\frac{5-y}{7} = m.$$

Then $$y = 5-7m.$$

Substituting in (1), $$7x+3(5-7m) = 50.$$

Whence, $$x = 5+3m.$$

Let $m = -1$. Then $x = 2$, and $y = 12$.

Let $m = 0$. Then $x = 5$, and $y = 5$.

Whence, $$x+y = 14 \text{ or } 10.$$

3. Let x = No. hogs, y = No. sheep, and z = No. lambs.

Then $$x+y+z=100, \tag{1}$$

and $$7x+3y+\tfrac{1}{2}z=100. \tag{2}$$

$(2)\times 2 =$ $$14x+6y+z=200.$$
$(1) =$ $$x+y+z=100.$$
Subtracting, $$13x+5y=100. \tag{3}$$

Whence, $$x=\frac{100-5y}{13} \text{ or } 7+\frac{9-5y}{13}.$$

$$\frac{9-5y}{13}\times 8=\frac{72-40y}{13} \text{ or } 5-3y+\frac{7-y}{13}.$$

Let $$\frac{7-y}{13}=m.$$

Then $$y=7-13m.$$

Substituting in (3), $$13x+5(7-13m)=100.$$

Whence, $$x=5+5m.$$

Substituting values of x and y in (1),

$$(5+5m)+(7-13m)+z=100.$$

Whence, $$z=88+8m.$$

Let $m=0$. Then $x=5$, $y=7$, and $z=88$.

4. Let x = numerators of fractions with denominator 12,

and y = numerators of fractions with denominator 18.

Then $$\frac{x}{12}+\frac{y}{18}=\frac{25}{36},$$

or, $$3x+2y=25. \tag{1}$$

Whence, $$x=\frac{25-2y}{3} \text{ or } 8+\frac{1-2y}{3}.$$

$$\frac{1-2y}{3}\times 2=\frac{2-4y}{3} \text{ or } y+\frac{2-y}{3}.$$

Let $$\frac{2-y}{3}=m.$$

Then $y = 2-3m.$

Substituting in (1), $3x+4-6m = 25.$

Whence, $x = 7-2m.$

Let $m = 0.$ Then $x = 7,$ and $y = 2.$

Let $m = -1.$ Then $x = 3,$ and $y = 5.$

Let $m = -2.$ Then $x = 3,$ and $y = 8.$

Let $m = -3.$ Then $x = 1,$ and $y = 11.$

Hence, there are 4 pairs of fractions.

5. Let $x =$ the number.

Then $\frac{x-6}{7}$, (1) $\frac{x-7}{8}$, (2) and $\frac{x-8}{9}$, (3) are integers.

Let $$\frac{x-6}{7} = m.$$

Then $$x = 6+7m. \tag{4}$$

Substituting in (2), $$\frac{(6+7m)-7}{8} \text{ or } \frac{7m-1}{8}. \tag{5}$$

Since $\frac{7m-1}{8}$ is integral, $\frac{7m-1}{8}\times 7$ is also integral.

$$\frac{7m-1}{8}\times 7 = \frac{49m-7}{8} \text{ or } 6m+\frac{m-7}{8}.$$

Let $$\frac{m-7}{8} = n, \quad \text{or,} \quad m = 7+8n. \tag{6}$$

Substituting in (4) the value of m in (6),

$$x = 6+7(7+8n) = 55+56n. \tag{7}$$

Substituting in (3) the value of x in (7),

$$\frac{(55+56n)-8}{9}, \text{ or } \frac{47+56n}{9}, \text{ or } 5+6n+\frac{2(1+n)}{9}.$$

Let $$\frac{1+n}{9} = r.$$

Then $$n = 9r-1. \tag{8}$$

Substituting in (7) the value of n in (8),

$$x = 55+56(9r-1),$$

or, $$x = 504r-1.$$

Making $r = 1$, $$x = 503.$$

6. Let x = number of men, and y = number of women.

Then $$19x+13y = 1000. \quad (1)$$

Whence, $$x = \frac{1000-13y}{19} \text{ or } 52+\frac{12-13y}{19}.$$

$$\frac{12-13y}{19}\times 3 = \frac{36-39y}{19} \text{ or } 1-2y+\frac{17-y}{19}.$$

Let $$\frac{17-y}{19} = m.$$

Then $$y = 17-19m.$$

Substituting in (1), $$19x+13(17-19m) = 1000.$$

Whence, $$x = 41+13m.$$

Let $m = 0$. Then $x+y = 58$.

Let $m = -1$. Then $x+y = 64$.

Let $m = -2$. Then $x+y = 70$.

Let $m = -3$. Then $x+y = 76$.

7. Let x = the first quotient.

Then $6(x)+5$ = the first part.

Let y = the second quotient.

Then $11(y)+4$ = the second part.

From these, $$6x+5+11y+4 = 200,$$

or, $$6x+11y = 191. \quad (1)$$

Whence, $$x = \frac{191-11y}{6} \text{ or } 31-y+\frac{5(1-y)}{6}.$$

Let $\dfrac{1-y}{6} = m.$

Then $y = 1-6m.$

Substituting in (1), $6x+11-66m = 191,$

or, $x = 11m+30.$

Let $m = 0.$ Then $x = 30$, and $y = 1.$

Whence, $6x+5 = 185,$ and $11y+4 = 15.$

Let $m = -1.$ Then $x = 19$, and $y = 7.$

Whence, $6x+5 = 119,$ and $11y+4 = 81.$

Let $m = -2.$ Then $x = 8$, and $y = 13.$

Whence, $6x+5 = 53,$ and $11y+4 = 147.$

8. Let $x =$ the number.

Then $\dfrac{x-13}{28}$, (1) $\dfrac{x-2}{19}$, (2) $\dfrac{x-7}{15}$, (3) are integers.

Let $\dfrac{x-13}{28} = m.$

Then $x = 13+28m.$ (4)

Substituting in (2) the value of x in (4),

$$\frac{13+28m-2}{19} = \frac{28m+11}{19} \text{ or } m+\frac{9m+11}{19}.$$

$$\frac{9m+11}{19}\times 17 = \frac{153m+187}{19} \text{ or } 8m+9+\frac{m+16}{19}. \quad (5)$$

Let $\dfrac{m+16}{19} = n.$

Then $m = 19n-16.$ (6)

Substituting value of m in (4), $x = 13+28(19n-16).$

Whence, $x = 532n-435.$ (7)

Substituting in (3) the value of x in (7),

$$\frac{(532n-435)-7}{15}. \quad (8)$$

$$\frac{532n-442}{15} = 35-29+\frac{7n-7}{15}.$$

Since $\frac{7(n-1)}{15}$ is integral, $\frac{n-1}{15}$ is integral.

Let $$\frac{n-1}{15} = r. \qquad (9)$$

Then, $$n = 15r+1.$$

Substituting in (7) the value of n in (9),

$$x = 532(15r+1)-435 \text{ or } 7980r+97.$$

Let $r = 0$. Then $$x = 97.$$

9. Let z = units' digit, y = tens' digit, and x = hundreds' digit.

Then $$x+y+z = 20, \qquad (1)$$

and $$\frac{(100x+10y+z)-16}{2} = 100z+10y+x,$$

or, $$98x-10y-199z = 16. \qquad (2)$$

$$\begin{array}{lrcl}
(1)\times 10 = & 10x+10y+\ \ 10z & = & 200. \\
(2) = & 98x-10y-199z & = & 16. \\ \hline
\text{Adding,} & 108x \qquad -189z & = & 216,
\end{array}$$

or, $$4x-7z = 8. \qquad (3)$$

Whence, $$x = \frac{8+7z}{4} \text{ or } 2+z+\frac{3z}{4}.$$

Since $\frac{3z}{4}$ is integral, $\frac{z}{4}$ is also integral.

Let $$\frac{z}{4} = m.$$

Then $$z = 4m.$$

Substituting in (3), $4x-28m = 8$.

Whence, $$x = 7m+2.$$

Substituting values of x and z in (1),

$$7m+2+y+4m = 20.$$

Whence, $y = 18 - 11m.$

Let $m = 1$. Then
$$x = 9,$$
$$y = 7,$$
$$z = 4,$$

and the number is 974.

10. Let x = number of horses, and y = number of oxen.

Then $$113x + 23y = 2500.$$

Whence, $$x = \frac{2500 - 23y}{113} \text{ or } 22 + \frac{14 - 23y}{113}.$$

$$\frac{14 - 23y}{113} \times 5 = \frac{70 - 115y}{113} \text{ or } y + \frac{2(35 - y)}{113}.$$

Let $$\frac{35 - y}{113} = m.$$

Then $$y = 35 - 113m.$$

Substituting in (1), $$113x + 23(35 - 113m) = 2500.$$

Whence, $$x = 23m + 15.$$

Let $m = 0$. Then $x = 15$, and $y = 35$.

11. Let x = first part, y = second part, and z = third part.

Then $$x + y + z = 30, \quad (1)$$

and $$7x + 19y + 38z = 745. \quad (2)$$

$$\begin{array}{ll}
(1) \times 38 = & 38x + 38y + 38z = 1140. \\
(2) = & 7x + 19y + 38z = 745. \\
\hline
\text{Subtracting,} & 31x + 19y = 395. \quad (3)
\end{array}$$

Whence, $$x = \frac{395 - 19y}{31} \text{ or } 12 + \frac{23 - 19y}{31}.$$

$$\frac{23 - 19y}{31} \times 2 = \frac{46 - 38y}{31} \text{ or } 1 - y + \frac{15 - 7y}{31}.$$

$$\frac{15-7y}{31}\times 9=\frac{135-63y}{31} \text{ or } 4-2y+\frac{11-y}{31}.$$

Let $\frac{11-y}{31}=m.$

Then $y=11-31m.$

Substituting in (3), $31x+19(11-31m)=395.$

Whence, $x=19m+6.$

Substituting values of x and y in (1),

$$19m+6+11-31m+z=30.$$

Whence, $z=12m+13.$

Let $m=0$. Then $x=6$, $y=11$, and $z=13$.

12. Let x = number of men,

y = number of women,

and z = number of children.

Then $x+y+z=30,$ (1)

and $\frac{7}{2}x+\frac{11}{8}y+\frac{1}{4}z=58,$

or, $28x+11y+2z=464.$ (2)

$$\begin{array}{ll} (1)\times 2 = & 2x+\ \ 2y+2z=\ \ 60. \\ (2) = & \underline{28x+11y+2z=464.} \\ \text{Subtracting,} & 26x+\ \ 9y \qquad =404. \end{array}$$ (3)

Whence, $x=\frac{404-9y}{26}$ or $15+\frac{14-9y}{26}.$

$$\frac{14-9y}{26}\times 3=\frac{42-27y}{26} \text{ or } 1-y+\frac{16-y}{26}.$$

Let $\frac{16-y}{26}=m.$

Then $y=16-26m.$

Substituting in (3), $26x+9(16-26m)=404.$

Whence, $x=9m+10.$

Substituting values of x and y in (1),

$$(9m+10)+(16-26m)+z = 30.$$

Whence, $$z = 4+15m.$$

Let $m = 0$. Then $$x = 10,\quad y = 16,\quad \text{and}\quad z = 4.$$

13. Let $x =$ the number of pages.

Then $$\frac{x-7}{11},\ (1)\quad \frac{x-3}{7},\ (2)\quad \text{and}\quad \frac{x-1}{5},\ (3)\quad \text{are integral.}$$

Let $$\frac{x-7}{11} = m.$$

Then $$x = 11m+7. \tag{4}$$

Substituting in (2) the value of x in (4), we have,

$$\frac{11m+4}{7} \text{ or } m+\frac{4(m+1)}{7}. \tag{5}$$

Let $$\frac{m+1}{7} = n.$$

Then $$m = 7n-1. \tag{6}$$

Substituting in (4) the value of m in (6),

$$x = 11(7n-1)+7 \text{ or } 77n-4. \tag{7}$$

Substituting in (3) the value of x in (7), we have,

$$\frac{77n-5}{5} \text{ or } 15n-\frac{2n+5}{5}. \tag{8}$$

$$\frac{2n+5}{5}\times 3 = \frac{6n+15}{5} \text{ or } n+3+\frac{n}{5}.$$

Let $$\frac{n}{5} = r.$$

Then $$n = 5r. \tag{9}$$

Substituting in (7) the value of n in (9),

$$x = 385r-4.$$

Let $r = 1$. Then $$x = 381.$$

14. Let $x =$ number of 25-cent pieces,

$y =$ number of 10-cent pieces,

and $z =$ number of 3-cent pieces.

Then $25x+10y+3z = 419.$ (1)

Whence, $z = \frac{419-25x-10y}{3}$ or $139-8x-3y+\frac{2-x-y}{3}.$

Let $\frac{2-x-y}{3} = m.$

Then $x = 2-y-3m.$ (2)

Substituting (2) in (1), $z = 123+5y+25m.$ (3)

Making $m = 0$, -1, etc., up to -12, and using all possible values of y, we obtain 102 sets of answers.

Making $m = -9$, $y = 22$, $x = 7$, and $z = 8$, the answers given.

15. Let $5x =$ number of dollars in A's fortune,

$7x =$ number of dollars in B's fortune,

and $9x =$ number of dollars in C's fortune.

Then $\frac{5x-1}{11}$, (1) $\frac{7x-2}{13}$, (2) and $\frac{9x-3}{15}$, (3) are integral.

$$\frac{5x-1}{11}\times 9 = \frac{45x-9}{11} \text{ or } 4x+\frac{x-9}{11}.$$

$$\frac{7x-2}{13}\times 2 = \frac{14x-4}{13} \text{ or } x+\frac{x-4}{13}.$$

$$\frac{9x-3}{15} = \frac{3x-1}{5}.$$

$$\frac{3x-1}{5}\times 2 = \frac{6x-2}{5} \text{ or } x+\frac{x-2}{5}.$$

If (1), (2), and (3) are integral,

$\frac{x-9}{11}$, (4) $\frac{x-4}{13}$, (5) and $\frac{x-2}{5}$, (6) are integral.

Let $$\frac{x-9}{11} = m.$$

Then $$x = 11m+9. \tag{7}$$

Substituting in (5) the value of x in (7), we have,

$$\frac{11m+5}{13}.$$

$$\frac{11m+5}{13} \times 6 = \frac{66m+30}{13} \text{ or } 5m+2+\frac{m+4}{13}.$$

Let $$\frac{m+4}{13} = n.$$

Then $$m = 13n-4. \tag{8}$$

Substituting value of m in (7),

$$x = 143n-35. \tag{9}$$

Substituting in (6) the value of x in (9), we have,

$$\frac{143n-37}{5}.$$

$$\frac{143n-37}{5} = 28n-7+\frac{3n-2}{5}.$$

$$\frac{3n-2}{5} \times 2 = \frac{6n-4}{5} \text{ or } n+\frac{n-4}{5}.$$

Let $$\frac{n-4}{5} = r.$$

Then $$n = 5r+4. \tag{10}$$

Substituting in (9) the value of n in (10),

$$x = 715r+537.$$

Let $r = 0$. Then $$x = 537.$$

Whence, $$5x = 2685,$$

$$7x = 3759,$$

and $$9x = 4833.$$

Art. 441.

1. $$m = 6.$$

Hence, $$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720.$$

$$m = 10.$$

Hence, $$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 3628800.$$

$$m = 8.$$

Hence, $$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320.$$

2. $$\text{Arrangements} = m(m-1)(m-2)\ldots\ldots(m-n+1).$$

$$m = 10, \qquad n = 3.$$

Hence, $$10 \times 9 \times 8 = 720.$$

$$\text{Combinations} = \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1 \times 2\ldots\ldots n}.$$

Hence, $$\frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120.$$

When $m = 10$, $$n = 4.$$

$$10 \times 9 \times 8 \times 7 = 5040,$$

and $$\frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = 210.$$

When $m = 10$, $$n = 5.$$

$$10 \times 9 \times 8 \times 7 \times 6 = 30240,$$

and $$\frac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5} = 252.$$

3. $$m = 12.$$

Hence, $$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 = 479001600.$$

4. $$\text{Combinations} = \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1 \times 2\ldots\ldots n}.$$

$$m = 15, \qquad n = 7.$$

Hence, $$\frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 6435.$$

5. $$m = 7.$$

Hence, $$1\times 2\times 3\times 4\times 5\times 6\times 7 = 5040.$$

6. $$\text{Combinations} = \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1\times 2\ldots\ldots n}$$

$$m = 8, \qquad n = 4.$$

Hence, $$\frac{8\times 7\times 6\times 5}{1\times 2\times 3\times 4} = 70.$$

$$\text{Arrangements} = m(m-1)(m-2)\ldots\ldots(m-n+1).$$

Hence, $$8\times 7\times 6\times 5 = 1680.$$

7. $$\text{Combinations} = \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1\times 2\ldots\ldots n}.$$

$$m = 25, \qquad n = 12.$$

Hence,

$$\frac{25\times 24\times 23\times 22\times 21\times 20\times 19\times 18\times 17\times 16\times 15\times 14}{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10\times 11\times 12} = 5200300.$$

8. $$m = 10.$$

Hence, $$1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10 = 3628800.$$

Art. 444.

2. Assume $$\frac{1+2x}{1-3x^2} = A+Bx+Cx^2+Dx^3+Ex^4+\text{etc.}$$

Clearing of fractions,

$$1+2x = A+Bx+\left\{\begin{matrix} C \\ -3A\end{matrix}\right\}x^2+\left\{\begin{matrix} D \\ -3B\end{matrix}\right\}x^3+\left\{\begin{matrix} E \\ -3C\end{matrix}\right\}x^4+\text{etc.}$$

Whence,

$$A = 1, \quad B = 2, \quad C-3A = 0, \quad D-3B = 0, \quad E-3C = 0.$$

Solving these, $A = 1$, $B = 2$, $C = 3$, $D = 6$, $E = 9$.

Substituting in the assumed equation,

$$\frac{1+2x}{1-3x^2} = 1+2x+3x^2+6x^3+9x^4+\text{etc.}$$

3. $$\frac{1-x}{2x^2+3x^3} = \frac{1}{x^2} \times \frac{1-x}{2+3x}.$$

Assume $$\frac{1-x}{2+3x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$$

Clearing of fractions,

$$1-x = 2A + \left\{\begin{matrix}2B\\3A\end{matrix}\right\}x + \left\{\begin{matrix}2C\\3B\end{matrix}\right\}x^2 + \left\{\begin{matrix}2D\\3C\end{matrix}\right\}x^3 + \text{etc.}$$

Whence, $A = \frac{1}{2}$, $2B+3A = -1$, $2C+3B = 0$, $2D+3C = 0$.

Solving these, $A = \frac{1}{2}$, $B = -\frac{5}{4}$, $C = \frac{15}{8}$, $D = -\frac{45}{16}$.

Substituting in the assumed equation,

$$\frac{1-x}{2+3x} = \tfrac{1}{2} - \tfrac{5}{4}x + \tfrac{15}{8}x^2 - \tfrac{45}{16}x^3 + \text{etc.},$$

and $$\frac{1-x}{2+3x} \times \frac{1}{x^2} = \frac{1}{2x^2} - \frac{5}{4x} + \frac{15}{8} - \frac{45}{16}x + \text{etc.}$$

4. Assume $$\frac{1}{1-2x+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$$

Clearing of fractions,

$$1 = A + \left\{\begin{matrix}B\\-2A\end{matrix}\right\}x + \left\{\begin{matrix}C\\-2B\\A\end{matrix}\right\}x^2 + \left\{\begin{matrix}D\\-2C\\B\end{matrix}\right\}x^3 + \left\{\begin{matrix}E\\-2D\\C\end{matrix}\right\}x^4 + \text{etc.}$$

Whence, $A = 1$, $B-2A = 0$, $C-2B+A = 0$,

$D-2C+B = 0$, $E-2D+C = 0$.

Solving these, $A = 1$, $B = 2$, $C = 3$, $D = 4$, $E = 5$.

Substituting in the assumed equation,

$$\frac{1}{1-2x+x^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \text{etc.}$$

5. $$\frac{2}{3x^2-2x^3} = \frac{1}{x^2} \times \frac{2}{3-2x}.$$

Assume $$\frac{2}{3-2x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}$$

Clearing of fractions,

$$2 = 3A + \left\{\begin{matrix} 3B \\ -2A \end{matrix}\right\} x + \left\{\begin{matrix} 3C \\ -2B \end{matrix}\right\} x^2 + \left\{\begin{matrix} 3D \\ -2C \end{matrix}\right\} x^3 + \left\{\begin{matrix} 3E \\ -2D \end{matrix}\right\} x^4 + \text{etc.}$$

Whence, $3A = 2,\qquad 3B-2A = 0,\qquad 3C-2B = 0,$

$$3D-2C = 0,\qquad 3E-2D = 0.$$

Solving these, $A = \frac{2}{3},\quad B = \frac{4}{9},\quad C = \frac{8}{27},\quad D = \frac{16}{81}.$

Substituting in assumed equation,

$$\frac{2}{3-2x} = \frac{2}{3} + \frac{4x}{9} + \frac{8x^2}{27} + \frac{16x^3}{81} + \text{etc.},$$

and $$\frac{2}{3-2x} \times \frac{1}{x^2} = \frac{2}{3x^2} + \frac{2^2}{3^2x} + \frac{2^3}{3^3} + \frac{2^4x}{3^4} + \frac{2^5x^2}{3^5} + \text{etc.}$$

APPENDIX.

Art. 198, *ex. 31*, p. 93.

80 ($5\frac{1}{4}$ oz.) = 420 oz. of copper;

and 20 ($4\frac{1}{4}$ oz.) = 85 oz. of tin.

Art. 217, *ex. 33*, p. 155.

Let u = the time required for all together.

Then $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{u}.$$

Substituting,

$$\frac{1}{8}+\frac{4}{15}+\frac{2}{9}=\frac{1}{u}.$$

$$\frac{37}{45}=\frac{1}{u}$$

$$37u=45$$

$$u=\frac{45}{37}, \text{ No. hrs.} = 1 \text{ hr. } 12\tfrac{36}{37} \text{ mi.}$$

***Art. 323**, ex. 10*, p. 243.

Solution for second answer in book.

$$\frac{2c^2}{d^2}+\frac{ac}{d}-(a-b)(2c+ad)\frac{x}{d}=(a+b)\frac{cx}{d}-(a^2-b^2)x^2.$$

Clearing of fractions,

$$2c^2+acd-(a-b)(2c+ad)\,dx=(a+b)\,cdx-(a^2-b^2)\,d^2x^2.$$

Transposing,

$$(a^2-b^2)(d^2x^2)-(a-b)(2c+ad)\;dx-(a+b)\,cdx=-2c^2-acd.$$

Dividing by a^2-b^2,

$$d^2x^2-\frac{(a-b)(2c+ad)+(a+b)c}{a^2-b^2}dx=-\frac{c(2c+ad)}{a^2-b^2}.$$

Completing the square,

$$d^2x^2-\frac{(a-b)(2c+ad)+(a+b)c}{a^2-b^2}dx+\left(\frac{(a-b)(2c+ad)+(a+b)c}{2(a^2-b^2)}\right)^2$$

$$=\frac{(a-b)^2(2c+ad)^2+2c(a^2-b^2)(2c+ad)+(a+b)^2c^2}{4(a^2-b^2)^2}$$

$$-\frac{4c(2c+ad)(a^2-b^2)}{4(a^2-b^2)^2}.$$

Uniting the fractions of the second member, it becomes,

$$\frac{(a-b)^2(2c+ad)^2-2c(a^2-b^2)(2c+ad)+(a+b)^2c^2}{4(a^2-b^2)^2}.$$

Extracting the square root of the equation,

$$dx - \frac{(a-b)\,(2c+ad)+(a+b)\,c}{2\,(a^2-b^2)} = \pm\frac{(a-b)\,(2c+ad)-(a+b)\,c}{2\,(a^2-b^2)}.$$

Whence,

$$dx = \frac{(a-b)\,(2c+ad)+(a+b)\,c}{2\,(a^2-b^2)} \pm \frac{(a-b)\,(2c+ad)-(a+b)\,c}{2\,(a^2-b^2)}$$

$$= \frac{2c+ad}{a+b}, \text{ or } \frac{c}{a-b};$$

and $x = \dfrac{2c+ad}{d\,(a+b)}$, or $\dfrac{c}{d\,(a-b)}$.

Art. 325, *ex. 20*, p. 244.

$$x^2 - \left(\frac{mn}{m+n} - \frac{m+n}{m-n}\right)x = \frac{mn}{m-n}.$$

By formula,

$$x = \frac{1}{2}\left(\frac{mn}{m+n} - \frac{m+n}{m-n}\right) \pm \sqrt{\frac{mn}{m-n} + \frac{1}{4}\left(\frac{mn}{m+n} - \frac{m+n}{m-n}\right)^2}.$$

Simplifying the radical,

$$\pm\sqrt{\frac{4\,mn}{4\,(m-n)} + \frac{1}{4}\left(\frac{mn}{m+n} - \frac{m+n}{m-n}\right)^2}$$

$$= \pm\frac{1}{2}\sqrt{\frac{4\,mn}{m-n} + \left(\frac{mn}{m+n} - \frac{m+n}{m-n}\right)^2}$$

$$= \pm\frac{1}{2}\sqrt{\frac{4\,mn\,(m+n)}{m^2-n^2} + \frac{m^2n^2}{(m+n)^2} - \frac{2\,mn\,(m+n)}{m^2-n^2} + \frac{(m+n)^2}{(m-n)^2}}$$

$$= \pm\frac{1}{2}\sqrt{\frac{m^2n^2}{(m+n)^2} + \frac{2\,mn\,(m+n)}{m^2-n^2} + \frac{(m+n)^2}{(m-n)^2}}$$

$$= \pm\frac{1}{2}\left(\frac{mn}{m+n} + \frac{m+n}{m-n}\right).$$

Substituting,

$$x = \frac{1}{2}\left(\frac{mn}{m+n} - \frac{m+n}{m-n}\right) \pm \frac{1}{2}\left(\frac{mn}{m+n} + \frac{m+n}{m-n}\right),$$

Whence,

$$x = \frac{mn}{m+n} \text{ or } -\frac{m+n}{m-n}.$$

Art. 415, *ex. 30*, p. 345.

Substituting the value of y in the equation $x = \frac{5y}{2}$,

$$x = \frac{5}{2}\left(-1 \pm \sqrt{-3}\right), \text{ or}$$

$$\frac{1}{2}\left(-5 \pm \sqrt{-75}\right).$$